Appendix A: *Relativistic Kinematics*

In this appendix we review some of the calculations required to solve various high-energy collision problems using *Special Relativity.*[1]

Contents

[1] Although several prominent theoretical physicists had developed the equations used by Einstein to describe the coordinate transformations of special relativity in the decades just preceding his seminal paper of 1905, it was his brilliant contribution to recognize that the theory must provide a *fundamental, kinematical description* of the behaviors of space and time themselves, independent of electromagnetism or any other dynamical (force-related) processes between material objects.

Some conventions regarding notation

Rest-mass, momentum, and energy will all be expressed in *energy units*, so that $\mathrm{mc}^2 \to m$ and $\vec{\mathrm{p}}\mathrm{c} \to \vec{p}$. In other words, we use the speed of light as our unit of velocity: $c \equiv 1$. Thus velocities will expressed as $\vec{\mathrm{v}}/\mathrm{c} \to \vec{v} = \vec{\beta}$ (the symbols v and β will be used interchangeably). Spatial positions will be expressed in units of time by dividing by *c*, so $\vec{\mathrm{x}}/\mathrm{c} \to \vec{x}$. If we need to express a quantity such as velocity, position, momentum, or mass in its actual units, then we use a symbol without italics: $\vec{\mathrm{x}}$, $\vec{\mathrm{v}}$, $\vec{\mathrm{p}}$, m, etc., as shown in the above expressions.

As illustrated by the examples above, spatial vectors will be denoted using an overscored arrow: $\vec{x}$. The 3 components of a spatial vector are labeled as:

$$\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z} \leftrightarrow (a_x, a_y, a_z) = (a_1, a_2, a_3)$$

Any *4-vectors* we use, such as the coordinates of a space-time event or a particle's 4-momentum, will be denoted using an italic symbol (usually uppercase) with an underscore: $\underline{X}$ or $\underline{P}$, for example, or as an ordered pair of a time-coordinate followed by a spatial vector, or as an ordered list of the time component followed by the 3 spatial components:

$$\underline{x} = (t, \vec{x}) = (x_0, \vec{x}) = (x_0, x_1, x_2, x_3)$$

Inertial observers' *reference frames* (space-time coordinate systems) will be denoted as $\mathbb{O}$ and $\mathbb{O}'$; 4-vector coordinates in these systems are $(t, \vec{x})$ and $(t', \vec{x}')$, respectively. The two systems' spatial axes are parallel and their origins coincide, so that $(t = 0, \vec{x} = 0)$ and $(t' = 0, \vec{x}' = 0)$ correspond to the same event. The relative velocity of $\mathbb{O}'$ with respect to $\mathbb{O}$ will normally be taken to be $\vec{\mathrm{v}}_{\mathbb{O}'} = \mathrm{v}\,\hat{x}$; obviously, $\mathbb{O}$ moves with velocity $-\mathrm{v}\,\hat{x}'$ in the $\mathbb{O}'$ coordinate system.

CONSERVED QUANTITIES AND LORENTZ INVARIANTS; REST-MASS

A dynamical quantity is *conserved* in a particular observer's reference frame if it doesn't vary with time. As in Newtonian mechanics, an isolated system's total energy *E* and linear momentum $\vec{p}$ are conserved quantities; thus so is the system's *4-momentum* $\underline{P} = (E, \vec{p})$. These quantities are also additive, so that the total energy and total momentum of a collection of particles is simply the sum of those of the individual particles:

$$\underline{P} = \Sigma \underline{P}_i = \left(\Sigma E_i, \Sigma \vec{p}_i\right) \tag{A-1}$$

A quantity is a *Lorentz invariant* (or, simply, an *invariant*)[2] if it has the same value in all inertial observers' reference frames (i.e. it is independent of the state of motion of an inertial

[2] The *Lorentz transformation* and *Lorentz invariance* were named by Henri Poincaré for the Dutch theoretical physicist Hendrik Antoon Lorentz, a towering figure in the development of the modern theories of classical electromagnetism and special relativity. Lorentz, along with George FitzGerald, Woldemar Voigt, and Joseph Larmor, developed the Lorentz contraction and time dilation equations during the period 1887-1900. Lorentz

observer). For example, the archetypal invariant is the *interval* between two space-time events:

$$(\Delta s)^2 \equiv (\Delta t)^2 - (\Delta\vec{x}\cdot\Delta\vec{x}) = (\Delta t')^2 - (\Delta\vec{x}'\cdot\Delta\vec{x}') \tag{A-2}$$

In (A-2) the 4-vectors $(\Delta t, \Delta\vec{x})$ and $(\Delta t', \Delta\vec{x}')$ denote the separation of the two events as observed in any two reference frames $\mathbb{O}$ and $\mathbb{O}'$. This property generalizes to the invariance of the scalar product of any two 4-vectors:

$$(a_0, \vec{a})\cdot(b_0, \vec{b}) \equiv a_0 b_0 - \vec{a}\cdot\vec{b} = (a_0', \vec{a}')\cdot(b_0', \vec{b}') \tag{A-3}$$

The *norm* of a 4-vector $\underline{A}$ is $\underline{A}\cdot\underline{A}$ and is a Lorentz invariant. If $\underline{A}\cdot\underline{A} > 0$, then $\underline{A}$ is called *time-like*; $\underline{A}\cdot\underline{A} < 0$ means $\underline{A}$ is *space-like*; a *light-like* 4-vector $\underline{A}$ has $\underline{A}\cdot\underline{A} = 0$. The interval $(\Delta s)^2$ is the norm of the space-time coordinate 4-vector $(\Delta t, \Delta\vec{x})$. A particle's *proper time* (the *interval* Δs between two events on its *world-line*) will be denoted as τ.

Lorentz transformation

Assume that some arbitrary 4-vector $\underline{A}$ has components $(a_0, \vec{a})$, with $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$, in the reference frame $\mathbb{O}$. In the frame $\mathbb{O}'$ $(\vec{v}_{\mathbb{O}'} = v\hat{x})$ the components of $\underline{A}$ are related to those in $\mathbb{O}$ by the *Lorentz transformation* $\Lambda(\vec{v}_{\mathbb{O}'})$: $(a_0', \vec{a}') = \Lambda(\vec{v}_{\mathbb{O}'})\cdot(a_0, \vec{a})$, defined so that:

Lorentz Transformation

$$\boxed{\begin{aligned} a_0' &= \gamma a_0 - \gamma\beta a_x \\ a_x' &= \gamma a_x - \gamma\beta a_0 \\ a_y' &= a_y; \quad a_z' = a_z \end{aligned}} \tag{A-4}$$

The *boost* $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}} \geq 1$. This transformation ensures that (A-3) is valid; it is analogous to a coordinate-system rotation in 3-dimensional Euclidean space which preserves the value of the scalar product $\vec{a}\cdot\vec{b}$ of two vectors in that space.

4-momentum and rest-mass

A particle or system's 4-momentum $\underline{P} = (E, \vec{p})$ is a 4-vector, and its norm $\underline{P}\cdot\underline{P}$ is a Lorentz invariant. This norm defines the system's *rest-mass* (or, more simply, *mass*), *m*:

$$m^2 \equiv \underline{P}\cdot\underline{P} = E^2 - p^2 \tag{A-5}$$

Our systems will always have real $m \geq 0$ ($\underline{P}$ is never space-like). If $m > 0$, then a reference frame can be found in which the system's spatial momentum vanishes $(p = 0)$. This

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was awarded the Nobel Prize in 1902 jointly with Pieter Zeeman for the original theory and later experimental observation of the *Zeeman Effect*.

reference frame is called the system's *center of momentum frame* or *rest frame*; in this frame $m = E$, or, using physical units, we get Einstein's famous $\mathrm{E} = \mathrm{mc}^2$.

> Because an isolated system's E and $\vec{p}$ are conserved, so is the system's rest-mass m, which is also a Lorentz invariant.

A *massless* particle, such as a photon, has $m = 0$, and $E = p$. Its 4-momentum is light-like, and its speed is c in all reference frames.

Consider a particle (or system) with $m > 0$. In its rest frame, the particle's 4-momentum is $\underline{P} = (m, 0)$. In another frame in which the particle has velocity $\vec{v}\ (= \vec{\beta})$, its 4-momentum components undergo the Lorentz transformation (A-4) and are given by:

$$\underline{P}(\vec{v}) = (E, \vec{p}) = (\gamma m, \gamma m\vec{v}) = m(\gamma, \gamma\vec{v}) = m\underline{U} \qquad \text{(A-6)}$$

The 4-vector $\underline{U} = (\gamma, \gamma\vec{v})$ is called the particle's *4-velocity*. Since a time interval $d\tau$ in its rest frame (an interval of the particle's *proper time*) is similarly boosted to the longer time interval $dt = \gamma d\tau$ in the frame within which it is moving (*time dilation*), we see that the 4-velocity is simply the rate of change of the particle's coordinate 4-vector $(t, \vec{x})$ with respect to its proper time: $\underline{U} = (dt/d\tau, d\vec{x}/d\tau)$. Since $\underline{P} \cdot \underline{P} = m^2$, it must be the case that $\underline{U} \cdot \underline{U} = 1$ (which you should confirm). From (A-6) we also get the useful result (remember, $v = \beta$):

$$\vec{p} = \vec{\beta} E \qquad \text{(A-7)}$$

which is also correct for massless particles such as photons (for which $v = 1$). Since $\beta \le 1$, then $p \le E$.

Kinetic energy

The *kinetic energy* T of a free particle (no externally-imposed potentials) is simply the difference between the particle's energy E and its mass m: $T = E - m$. Using equation (A-6) we see that

$$T = (\gamma - 1)m \qquad \text{(A-8)}$$

A relation between p and T analogous to the Newtonian $T = p^2/2m$ may be derived using (A-5), realizing that $E^2 - m^2 = (E - m)(E + m) = T(T + 2m)$:

$$T = \frac{p^2}{2m + T} \qquad \text{(A-9)}$$

In the nonrelativistic limit $T \ll m$, and the expression (A-9) approaches the Newtonian one.

SOME EFFECTS OF REFERENCE FRAME CHANGES

Doppler shift

A free particle with well-defined 4-momentum $\underline{P}$ may be represented quantum-mechanically as a plane-wave with frequency $\omega = E/\hbar$ and wave vector $\vec{k} = \vec{p}/\hbar$. Thus $\underline{K} \equiv (\omega, \vec{k}) = \underline{P}/\hbar$ is a 4-vector. The phase ϕ of the plane wave at any space-time location $\underline{X} = (t, \vec{x})$ is then

$$\phi = -(\omega, \vec{k}) \cdot (t, \vec{x}) = -\underline{K} \cdot \underline{X} \tag{A-10}$$

and is a Lorentz invariant. Changing reference frames will result in a Lorentz transformation of the components of the wave's 4-vector $\underline{K}$: this is the origin of the *Doppler shift* [3] of the wave frequency and wavelength, as well as a change in its direction of propagation (see the next section).

Assume the new reference frame moves with velocity $\vec{\beta}$; the component of $\vec{k}$ parallel to $\vec{\beta}$ is then $k_{\|} = \vec{k} \cdot \vec{\beta}/\beta = k\cos\theta$, where θ is the angle between $\vec{k}$ and $\vec{\beta}$. From the Lorentz transformation (A-4) the wave frequency ω' (time-like component of $\underline{K}'$) in the moving frame is:

Doppler shift formula

$$\boxed{\omega' = \gamma(\omega - \beta k_{\|}) = \gamma(\omega - \vec{\beta} \cdot \vec{k})} \tag{A-11}$$

In the case of a photon $\omega = k$, and the Doppler shift formula is $\omega'/\omega = \gamma(1 - \beta\cos\theta)$.

Relativistic beaming

As mentioned above, the observed propagation direction of a photon depends on the state of motion of the reference frame. Because of the Earth's orbital motion, for example, this effect is the primary cause of the so-called *stellar aberration* (coordinated, cyclical shifts in the positions of distant astronomical objects).[4] Another important example is *relativistic beaming* of the radiation emitted by a high-speed source toward its direction of motion. This is the effect we shall now investigate by continuing the discussion of the previous section.

A photon propagates with wave vector $\underline{K}' = (k', \vec{k}')$ in reference frame $\mathbb{O}'$, which moves with velocity $\vec{v}_{\mathbb{O}'} = \beta\hat{x}$ relative to the frame $\mathbb{O}$ (remember, for a photon $\omega = k$). Let $\vec{k}' = k'_x\hat{x}' + k'_y\hat{y}'$. Then the Lorentz transformation (A-4) of this wave vector to the frame $\mathbb{O}$ is:

$$k = \gamma k' + \gamma\beta k'_x \qquad k_x = \gamma k'_x + \gamma\beta k' \qquad k_y = k'_y$$

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[3] The Doppler Effect is named for the Austrian physicist Christian Doppler, who published his theory of the shift in the observed wavelength of light caused by relative motion of source and observer in 1842. His nonrelativistic theory was equivalent to (A-11) in the limit that $\gamma \to 1$.

[4] Stellar aberration, first noticed by astronomers in the late 17th century, was systematically measured and explained by the English astronomer James Bradley in 1729 as due to the finite speed of light.

Let θ' represent the angle of the photon's path in frame $\mathcal{O}'$ with respect to the frame's direction of motion. Then $\cos\theta' = k'_x/k'$ (and so on for $\sin\theta'$ and $\tan\theta'$). The corresponding angle θ of the photon's path in frame $\mathcal{O}$ may be determined in any of a number of ways:

$$\begin{aligned}\cos\theta &= \frac{k_x}{k} = \frac{\gamma k'_x + \gamma\beta k'}{\gamma k' + \gamma\beta k'_x} = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \\ \tan\theta &= \frac{k_y}{k_x} = \frac{k'_y}{\gamma k'_x + \gamma\beta k'} = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}\end{aligned} \tag{A-12}$$

Since $0 < \theta' < \pi$ and $\gamma > 1$, then if $\beta > 0$ the second of equations (A-12) clearly implies that $\theta < \theta'$, as it should be: the photon picks up an extra bit of velocity parallel to $\vec{\beta}$ in frame $\mathcal{O}$ because of the relative motion of $\mathcal{O}'$. For example, the first of equations (A-12) demonstrates that radiation emitted perpendicular to its motion in a source's rest frame $(\cos\theta' = 0)$, will propagate at the acute angle given by $\theta = \arccos\beta$ in the observer's frame; for large $\beta \sim 1$, $\theta \approx \sqrt{2(1-\beta)}$. Clearly, this effect is dramatic for $\gamma \gg 1$: the radiation from such a source is nearly all confined to a cone about the direction of the source's velocity vector (as well as being Doppler-shifted to much shorter wavelengths within this cone).

Motion of the center of momentum frame

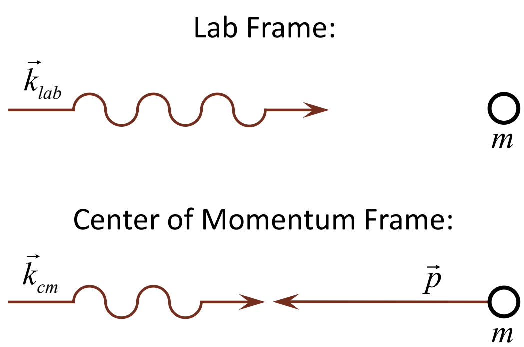

Figure A-1: 4-momentum diagrams of the lab and center of momentum (CM) frames for a system comprised of a photon approaching a stationary particle.

Assume a photon with energy k_{lab} approaches a stationary object (such as a single electron) with mass m. We wish to calculate the energy of the photon as observed in this system's *center of momentum* (CM) frame (k_{cm}) and the speed β_{cm} of this frame with respect to the lab frame. The situation is illustrated in Figure A-1.

In the lab frame the system has total 4-momentum $\underline{P}_{lab} = (k_{lab} + m,\ \vec{k}_{lab})$; relation (A-7) then implies that $k_{lab} = \beta_{cm}(k_{lab} + m)$, so

$$\beta_{cm} = \frac{1}{1 + (m/k_{lab})} \tag{A-13}$$

β_{cm} is, of course, also the speed of the particle m as observed in the CM frame. Using (A-13) in the Doppler formula (A-11), with $\vec{\beta}_{cm} \parallel \vec{k}_{lab}$,

$$k_{cm} = \gamma_{cm}(1-\beta_{cm})\,k_{lab}$$

$$\text{and} \qquad \gamma_{cm}(1-\beta_{cm}) = \frac{1-\beta_{cm}}{\sqrt{1-\beta_{cm}{}^2}} = \sqrt{\frac{(1-\beta_{cm})^2}{1-\beta_{cm}{}^2}} = \sqrt{\frac{1-\beta_{cm}}{1+\beta_{cm}}}$$

$$\therefore \quad \frac{k_{cm}}{k_{lab}} = \frac{1}{\sqrt{1+\,2(k_{lab}/m)}} \tag{A-14}$$

For example, if the photon is a gamma ray resulting from $^{137}\mathrm{Cs}$ decay $(k_{lab} = 0.662\,\mathrm{MeV})$ and the target particle is an electron $(m = 0.511\,\mathrm{MeV})$, then $\beta_{cm} = 0.56$, $\gamma_{cm} = 1.2$, and $k_{cm} = 0.349\,\mathrm{MeV}$.

COLLISIONS INVOLVING PHOTONS

Photoelectric absorption and *emission*, *pair production*, and *Compton scattering* are among the important processes whose kinematics can be analyzed as collisions between photons and other particles, which we now consider.

Photoelectric absorption and emission

Figure A-2: Photoelectric absorption of a photon by a system with initial mass M (shown in the lab frame). The photon is destroyed by the interaction, and the mass of the target system is increased by the collision to the new value $M^*>M$. Conservation of momentum requires that the product M^* recoil with the incoming photon's momentum, requiring some of the incoming photon's energy.

A photon may undergo an *inelastic* collision with a target system only if the target has some sort of internal degree of freedom which can absorb the energy of the collision, as illustrated in Figure A-2. This figure shows the collision in the lab frame where the target is initially at rest; the photon is destroyed by the collision. As we will quickly see, conservation of 4-momentum by the process requires a target mass increase: $\Delta E = M^* - M > 0$. This is possible for a complicated target object such as an atom or nucleus, which can enter an excited energy state by a suitable rearrangement of it several constituent parts (electrons or nucleons).

The total 4-momentum before and after the collision: $\underline{P} = (k + M, \vec{k}) = (E_{M*}, \vec{k})$. Using (A-5), $(k+M)^2 = E_{M*}^2 = (M*)^2 + k^2 = (M + \Delta E)^2 + k^2$. Solving for k:

Photoelectric Absorption:

$$k = \Delta E\left(1 + \frac{\Delta E}{2M}\right); \qquad \Delta E \equiv M* - M \tag{A-15}$$

Since $k > 0$, so is ΔE; from the above expression, $k > \Delta E$. This makes sense, because some of the photon's energy must be used to supply the recoil kinetic energy of $M*$. Note that *there must be a mass increase* of the target for the photon to be absorbed, otherwise 4-momentum cannot be conserved by the absorption; thus, an elementary particle such as an electron cannot absorb a photon, because it does not have any internal excited states to which it can transition and provide for an increase in its mass.

Photoelectric emission is the opposite process: an excited system (atom or nucleus) decays to a lower energy state by emitting a photon, as shown in Figure A-3.

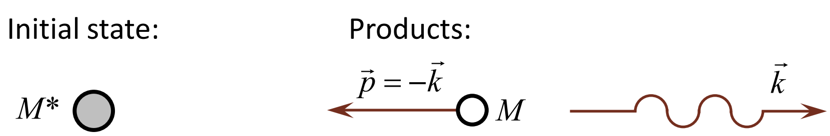

Figure A-3: Photoelectric emission of a photon by an initially motionless system in an excited state with initial mass $M*$. The mass of the system decreases to M as it decays to a lower energy state by emitting the photon. Because conservation of momentum requires that the system recoil, not all of the mass loss can go into the outgoing photon energy.

Again, we equate the 4-momenta before and after the emission and solve for the photon energy k: $(M*, 0) = (E_M + k, \vec{p} + \vec{k})$; $(M* - k)^2 = E_M^2 = (M* - \Delta E)^2 + k^2$. Thus:

Photoelectric Emission:

$$k = \Delta E\left(1 - \frac{\Delta E}{2M*}\right); \qquad \Delta E \equiv M* - M \tag{A-16}$$

Thus $k < \Delta E$, because some energy is lost to provide for the recoil momentum of M. Note that this process is simply the reverse of that of photoelectric absorption, but observed in the CM frame rather than the lab frame (compare Figure A-3 with Figure A-2).

The target particle M in the photoelectric processes (A-15) or (A-16) is usually an atom or nucleus, and the fractional corrections to the approximate relation $k \approx \Delta E$ are quite small and in most cases completely negligible. Since an *atomic mass unit* corresponds to a rest energy of $\sim 900\,\text{MeV}$, M is nearly always many thousands of times larger than the k and ΔE values involved in our experiments, so $k \approx \Delta E$ is a very good approximation indeed. The truly dramatic exceptions to this observation are experiments involving the Mössbauer effect, which can measure gamma-ray fractional energy shifts smaller than 10^{-11}; in this case $\Delta E/M$ would be many orders of magnitude larger than these shifts if M were to consist of only a single atom.

Pair production

An interesting and important cousin of photoelectric absorption is *pair production*, in which a high-energy photon can be absorbed in a collision with a charged particle and produce a charged *particle-antiparticle pair*, as illustrated in Figure A-4.

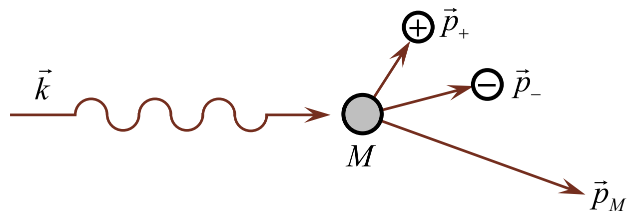

Figure A-4: Pair production by a photon collision with the charged target particle *M*. If the incoming photon energy is great enough, a particle-antiparticle pair may be produced as the photon is absorbed.

Various conservation laws require that this process produce a particle-antiparticle pair as opposed to a single new particle. The particles produced carry electromagnetic charge (with opposite signs) because the photon is the quantum carrier of electromagnetic force. The target particle must also be charged (e.g. an atomic nucleus) so that the photon can interact strongly with it.

We analyze the kinematics of this process in the same way as for photoelectric absorption, equation (A-15). The target particle has mass M; the product "particle" is the system consisting of the original particle and the new particle-antiparticle pair. The rest mass M^* of this assemblage is the total energy of the 3 particles as seen in its rest frame following the absorption, which is, of course, the CM frame of the collision (momentum conservation!). The minimum value for M^* would be the sum of the three particles' masses; this would occur only when they are all at rest in the CM frame following the collision. Since the antiparticle has the same mass as its particle counterpart, the minimum value for M^* is given by $M^* = M + 2m$, where m is the mass of the particle produced.

With this minimum value for M^*, equation (A-15) provides the minimum photon energy required for pair production:

Pair production threshold: $$k_{min} = 2m\left(1 + \frac{m}{M}\right) \tag{A-17}$$

The lightest charged particle is the electron $(m = 0.511\,\text{MeV})$; the threshold energy for electron-positron pair production whenever a photon interacts with an atomic nucleus (or even a lone proton, so that $M \gg m$) is just over $2m = 1.02\,\text{MeV}$.[5] The probability of pair production rises rapidly with increasing photon energy above this threshold – pair production becomes the dominant interaction above $\sim 10\,\text{MeV}$. Above $150\,\text{MeV}$ or so other, heavier particles are produced, such as π mesons (light quark-antiquark bound pairs with "2m" $\approx 140\,\text{MeV}$) and muon-antimuon pairs $(m = 106\,\text{MeV})$.

Compton scattering

We shall analyze the kinematics of Compton scattering [6] as an elastic collision between a photon and a charged particle (initially at rest) as shown in Figure A-5. Given the incoming photon's energy k_0 and particle mass m, we want the outgoing photon energy k as a function of its outgoing direction (turned by angle θ from the incoming photon trajectory); the target particle's outgoing 4-momentum following the interaction is $(E_m, \vec{p})$.

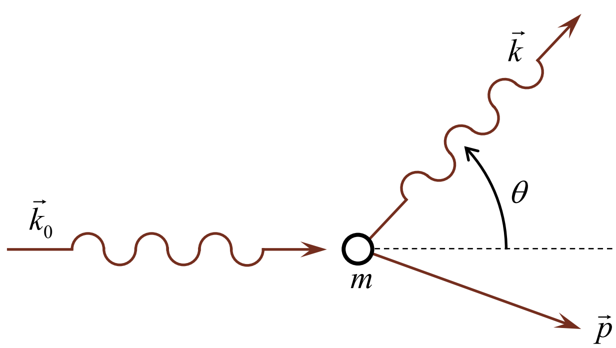

Figure A-5: Compton scattering of a photon by an initially stationary particle.

Start with energy conservation: the total system energy is unaffected by the collision. Since the collision is elastic, the particle's mass is unaffected by it. Thus:

$$\begin{aligned} k_0 + m &= k + E_m \\ (k_0 + m - k)^2 &= E_m{}^2 = p^2 + m^2 \end{aligned}$$

Now use momentum conservation to determine p^2, which is the sum of the squares of its horizontal and vertical components in Figure A-5. The vertical component equals that of the

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[5] Physicist Carl Anderson discovered the positron at Caltech in 1932 by observing cosmic rays in a cloud chamber; his discovery earned him the 1936 Nobel Prize in physics. British physicist Patrick Blackett first identified electron-positron pair production in 1933, also by observing cosmic rays. He earned the 1948 Nobel Prize for his instrument development and discoveries.

[6] The American physicist Arthur Compton discovered the effect named for him in 1922. His paper of 1923 included the equivalent of equation (A-18). His discovery and correct interpretation of its meaning earned Compton the Nobel Prize in physics in 1927.

outgoing photon: $k\sin\theta$; the horizontal component of p equals the difference between two photons' horizontal components: $k_0 - k\cos\theta$.

$$p^2 = (k_0 - k\cos\theta)^2 + (k\sin\theta)^2$$
$$= k_0{}^2 + k^2 - 2k_0 k\cos\theta$$

Substituting into the energy conservation expression and collecting terms:

$$m(k_0 - k) = k_0 k(1+\cos\theta)$$

Solving for k gives the important result:

Compton scattering formula

$$\boxed{k = \frac{k_0}{1 + \frac{k_0}{m}\left(1-\cos\theta\right)}} \qquad \text{(A-18)}$$

INELASTIC COLLISIONS OF IDENTICAL PARTICLES

Finally, we contrast the inelastic collision of a pair of identical particles in the lab frame (one particle stationary) and the center of momentum frame (both particles moving toward each other). The energy of the collision, if high enough, can result in the creation of many particle-antiparticle pairs — including rare, massive ones important for fundamental research into the nature of elementary particles and the forces between them. This discussion will indicate why modern accelerators for high-energy research (such as the *Large Hadron Collider*, or *LHC*) are often designed to use *colliding beams* rather than stationary targets.

Assume that two identical particles each of mass M inelastically collide to form a single object of mass M^*. The energy difference $\Delta E = M^* - 2M$ may then be available to create new particles as a result of the collision, as in the *Pair production* process discussed previously. In the case of two colliding beams (where the lab frame is also the center of momentum frame), the expression for ΔE is trivial: $\Delta E = 2T$, where T is the kinetic energy of each particle (Figure A-6).

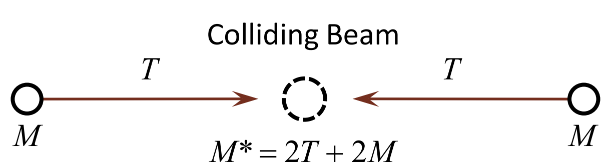

Figure A-6: Colliding beam energetics. Each particle arrives at the collision point with kinetic energy T, so $2T$ is the additional energy available for particle creation following the collision.

The case of a stationary target particle is slightly more difficult to analyze, but the algebra is straightforward. The inbound particle again has kinetic energy T, but the target particle is at rest. Because the daughter particle must recoil following the collision (momentum conservation!), some of the kinetic energy T will be required for this recoil, as was seen in the case of a photon absorption [Figure A-2 and equation (A-15)]. Consider the nonrelativistic case first: $T \ll M$ as shown in Figure A-7.

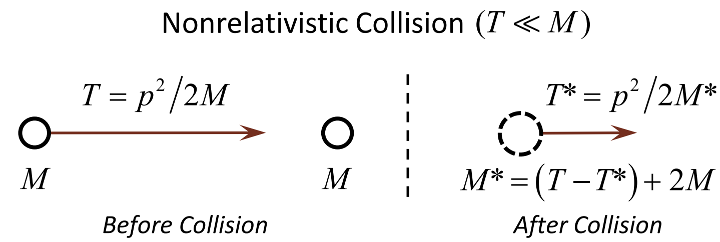

Figure A-7: Nonrelativistic particle colliding with a fixed target. Since the daughter particle mass $M^* \approx 2M$, the recoil kinetic energy is $T^* \approx T/2$. Thus, only half of the incoming particle's kinetic energy remains for particle creation, 1/4 of that available in the colliding beam arrangement (Figure A-6).

Because the incoming particle is nonrelativistic, its momentum is given by the Newtonian expression $p = \sqrt{2MT}$. This must, of course, match the recoil momentum following the collision, so that $p^2/2 = M\,T = M^* T^*$. But since $T \ll M$ and therefore $T^* \ll M$, it must be the case that $M^* \approx 2M$ to a high degree of accuracy, and therefore $T^* = T/2$. So the energy available for particle creation is only:

Nonrelativistic, single-beam energy: $\boxed{\Delta E = M^* - 2M = T/2}$ (A-19)

This is just 1/4 of that available in the colliding beam arrangement of Figure A-6.

Now assume that the incoming particle beam is quite relativistic, so that $T = (\gamma - 1)M \gg M$. The incoming particle's total energy $E = \gamma M$, and its momentum, from equation (A-7), is $p = \beta E$ (Figure A-8).

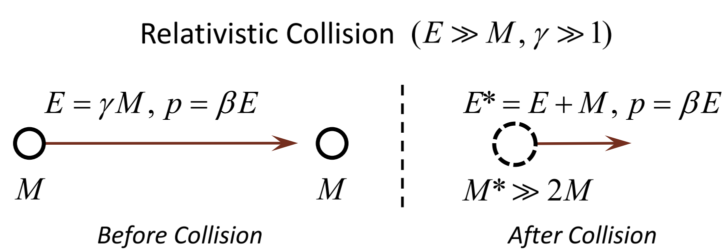

Figure A-8: Relativistic particle colliding with a fixed target.

Thus the total 4-momentum of the system before the collision is

$$\underline{P} = (E, \vec{\beta}E) + (M, 0) = (E + M, \vec{\beta}E)$$

The total rest-mass of this pre-collision system is given by the norm of its 4-momentum: $m^2 = \underline{P} \cdot \underline{P}$, equation (A-5), and this must equal the daughter system's rest-mass M^*. Thus:

$$\begin{aligned}(M^*)^2 &= \underline{P} \cdot \underline{P} = (E+M)^2 - \beta^2 E^2 \\ &= (1-\beta^2)E^2 + M^2 + 2ME\end{aligned}$$

But $1-\beta^2 = \gamma^{-2}$, and $M = E/\gamma$. Therefore:

$$(M^*)^2 = \frac{E^2}{\gamma^2} + \frac{E^2}{\gamma^2} + 2\frac{E^2}{\gamma} = \frac{E^2}{\gamma/2}\left(1 + \frac{1}{\gamma}\right)$$

So for a highly relativistic incoming particle beam ($\gamma \gg 1$ so that $E \approx T$ and $\Delta E = M^* - 2M \approx M^*$),

Relativistic, single-beam energy:

$$\boxed{\Delta E = M^* - 2M \approx \frac{T}{\sqrt{\gamma/2}}} \qquad \text{(A-20)}$$

This result is a factor of $1/\sqrt{2\gamma}$ *smaller* than the $\Delta E = 2T$ available in the colliding beam arrangement of Figure A-6. For high-energy accelerators, this factor is much more dramatic than the classical limit of 1/4. In the case of the *LHC*, where protons are accelerated to $E = 5\,\text{TeV}$ (or more) and then collided, $\gamma \approx 5300$. The colliding beam arrangement of the *LHC* results in collision energies over 100 times greater than what would be possible using a fixed target!