

## Experiment 33

## THE STERN-GERLACH EXPERIMENT

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## INTRODUCTION

In this experiment you will perform a version of the famous experiment of Otto Stern and Walther Gerlach. You will observe the quantization of a vector component of the magnetic moment of the electron, measure the value of the *Bohr magneton*,  $\mu_B$ , and consequently determine an estimate of *Planck's Constant*,  $\hbar$ .

After two years of effort, the *Stern-Gerlach* experiment was finally performed successfully in early 1922 in Frankfurt, Germany, using a beam of neutral silver atoms (Figure 1). It was meant to be a test of the early atomic quantum theory of Niels Bohr, Arnold Sommerfeld, and Peter Debye. The quantization conditions of this theory implied that the outer electron's "orbit" in hydrogen-like atoms should be able to assume only two discrete orientations with respect to an external magnetic field, a condition which was referred to as *space quantization* (reference 1). The resultant component of the magnetic moment of the atom due to the little current loop formed by the electron's orbital motion should assume only two values,  $\pm\mu_B$ .

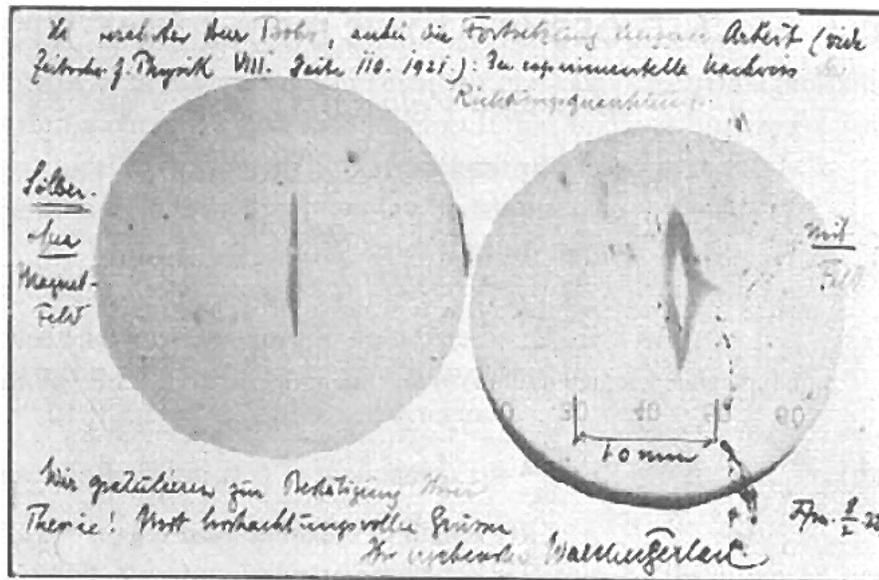


Figure 1: Gerlach's February 1922 postcard to Niels Bohr showing an image of his result. Note that the splitting of the silver atom beam was only a fraction of a millimeter (right-hand image).

Contrary to their understanding of the experiment, however, Stern and Gerlach's results did not actually provide a confirmation of the early quantum theory at all, but instead observed the quantization of the inherent angular momentum of the electron itself, now known as *spin*. As pointed out in reference (1), the agreement of the Stern-Gerlach experiment with the old quantum theory was purely a coincidence, since the silver atom in its ground state actually has zero orbital angular momentum. The total spin angular momentum of the electron (and all other elementary material particles, known collectively as *Fermions*) has a quantum number of  $\frac{1}{2}$ , and can therefore only assume two "orientations" with respect to an externally-defined direction. The resultant magnetic moment of the electron likewise has only two possible component values

along an externally-defined vector direction, and it is this magnetic moment that the experiment determines. Because the *gyromagnetic ratio* of the electron,  $g_e$ , is almost exactly 2, the component of electron magnetic moment along any direction also assumes values of very nearly  $\pm\mu_B$ , which naturally led to confusion in interpreting the results. In fact, it wasn't until 1927, two years after the postulation of electron spin in 1925, that the results of the Stern-Gerlach experiment were correctly interpreted as due to electron spin.

You will perform the experiment using a beam of neutral potassium atoms, whose magnetic moment is dominated by that of the spin of the alkali metal's single valence electron. By passing a beam of these atoms through a magnetic field with a uniform, nonzero gradient perpendicular to the beam axis, the atoms are deflected by the force exerted by the field on the atoms' magnetic dipole moment. By observing the deflection of the atoms from the beam axis, you will analyze the nature of this dipole moment, which is a consequence of the electron's angular momentum.

#### THEORY: A QUANTUM MAGNETIC DIPOLE IN AN EXTERNAL FIELD

Magnetic dipole moments of atomic and subatomic systems arise as a consequence of the angular momenta of the systems, so to understand the behavior of the magnetic moments we must understand angular momentum. In this experiment, for example, the magnetic dipole moment of the potassium atom is a consequence of the angular momentum of its single valence electron. The intrinsic angular momentum of the charged electron gives rise to its magnetic dipole moment, and as a result the dipole moment vector is aligned with the electron's angular momentum vector. Since the electron has no internal degrees of freedom which can dissipate energy, the torque exerted by an external magnetic field on the dipole moment would be expected to cause the electron to precess about the field direction, and the angle the magnetic moment  $\vec{\mu}$  makes with the field would remain constant. Interestingly and contrary to the predictions of classical theory, the electron does not radiate electromagnetic energy continuously as a result of this precession; such radiation would cause the electron's magnetic moment to eventually align with the field, as it would provide a means for the electron to lose this potential energy due to the relative alignment of the dipole moment vector and the external field.

The Caltech student will get a thorough introduction to the quantum theory of angular momentum in Physics 125. A good text providing this introduction is reference (2). The quantum theory requires that the squared magnitude of any angular momentum be quantized as:

$$J^2 = j(j+1)\hbar^2; \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \quad (1)$$

As indicated in equation (1), the total angular momentum quantum number  $j$  must be a nonnegative integer multiple of  $\frac{1}{2}$ . If the angular momentum is due to the motion of one or more masses about some spatial reference point (as in classical physics), then it turns out that the quantum number  $j$  must be an integer; in this case it is called *orbital angular momentum* and is usually referred to as  $L$  rather than  $J$ . Half-integer values of  $j$  cannot result from motion alone,

but must include some form of *intrinsic angular momentum* of a most elemental nature. Such a value of angular momentum is called *spin* (although no bits of matter inside an elementary particle are in motion), and is referred to as  $S$ . Generally, the total angular momentum  $\vec{J}$  of a system will be some quantum-mechanical vector combination of  $L$  and  $S$ . The intrinsic angular momentum quantum number of all material elementary particles (such as the electron) seems to be  $s = \frac{1}{2}$ .

Given an angular momentum vector with squared magnitude  $J^2$ , quantum mechanics permits the determination of only one component of the vector in a single experiment (which is conventionally chosen to be in the direction  $\hat{z}$ ):  $J_z$ . Given total angular momentum quantum number  $j$ , the quantization of  $J_z$  is:

$$J_z = m\hbar; \quad m = -j, -j+1, \dots, j-1, j \quad (2)$$

In the case of an electron's spin, this would imply that the only values of  $S_z = \vec{S} \cdot \hat{z}$  are  $S_z = \pm \frac{1}{2}\hbar$ . The quantum number  $m$  is often called the *magnetic quantum number*.

This quantum phenomenon which allows only a finite number of values for the component of angular momentum along any externally-defined direction is the modern definition of *space quantization*.

The magnetic moment of an atomic electron due to its orbital angular momentum is given by the expression:

$$\text{Orbital magnetic moment: } \vec{\mu} = \frac{e}{2m_e} \vec{L}; \quad \mu_z = \frac{e\hbar}{2m_e} m_L = \mu_B m_L \quad (3)$$

where  $m_L$  is the magnetic quantum number, as in equation (2), of the orbital angular momentum vector  $\vec{L}$  (recall that  $m_L$  must be an integer, since this is orbital angular momentum). The Bohr magneton,  $\mu_B = 0.927 \times 10^{-23}$  Joule/Tesla, is the fundamental quantum of magnetic moment. More generally, for any quantum system in a state with total angular momentum  $\vec{J} = \vec{L} + \vec{S}$ , we find that the  $\hat{z}$  component of the magnetic moment is given by

$$\mu_z = g\mu_B m \quad (4)$$

where the gyromagnetic ratio  $g$  is a constant independent of the magnetic quantum number  $m$ , but usually depends on the total angular momentum quantum number  $j$  ( $g$  need not be an integer or half-integer, although  $m$  must be). In the case of the electron, the relativistic effect called *Thomas precession* requires that  $g_e = 2$  (exactly). The quantum electrodynamics of Feynman provides higher-order corrections which increase its value by about 1% (the so-called *anomalous magnetic moment* of the electron). The theoretical prediction of the value of  $g_e - 2$  is the most precise numerical prediction of any physical theory to date.

A magnetic dipole with vector magnetic moment  $\vec{\mu}$  in an external magnetic field  $\vec{B}$  has potential energy

$$U = -\vec{\mu} \cdot \vec{B} \quad (5)$$

Since only  $\vec{B}$  represents a field with a gradient, and the curl of  $\vec{B}$  vanishes in empty space, the force (not the torque) on a magnetic dipole is

$$\begin{aligned} \vec{F} &= -\nabla U = \nabla(\vec{\mu} \cdot \vec{B}) = (\vec{\mu} \cdot \nabla)\vec{B} + \vec{\mu} \times (\nabla \times \vec{B}) \\ \therefore \vec{F} &= \mu_x \frac{\partial \vec{B}}{\partial x} + \mu_y \frac{\partial \vec{B}}{\partial y} + \mu_z \frac{\partial \vec{B}}{\partial z} \end{aligned} \quad (6)$$

In the case of a quantum system we could use equation (4) to calculate the components of  $\vec{\mu}$ ; the force would become:

$$\vec{F} = g\mu_B \left[ m_x \frac{\partial \vec{B}}{\partial x} + m_y \frac{\partial \vec{B}}{\partial y} + m_z \frac{\partial \vec{B}}{\partial z} \right] \quad (\text{Wrong!}) \quad (7)$$

Unfortunately, this equation is useless for a quantum mechanical system (thus the “Wrong”)! It turns out to be impossible to simultaneously know the values of all components of a quantum system’s angular momentum (and thus its magnetic moment) – in fact, **only one component** may be measured as long as the magnitude of the total angular momentum is also fixed – that’s why there is only one magnetic quantum number  $m$  (and not  $m_x$ ,  $m_y$ ,  $m_z$  as in equation (7)). The resolution of this problem is not too hard, however: we just use the direction of  $\vec{B}$ , and realize that the force on our quantum magnetic dipole must be along this direction.

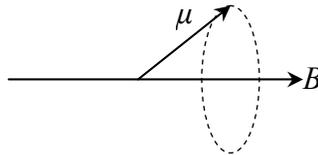


Figure 2: A magnetic field generates a torque on a magnetic dipole, which causes the dipole to precess about the field direction, like a gyroscope. The dipole is a manifestation of the angular momentum vector of the system.

We can most easily see this fact by thinking of the situation for a classical dipole whose moment is parallel to the angular momentum of a classical system. The  $\vec{B}$  field exerts a torque on the dipole, which causes the dipole moment vector to precess about the magnetic field vector direction, as in Figure 2, because the dipole is associated with the angular momentum of the object. Clearly, in this case the component of  $\vec{\mu}$  perpendicular to  $\vec{B}$  will average to zero (and thus the average force perpendicular to  $\vec{B}$  from equation (6)), whereas the component parallel to  $\vec{B}$  remains constant. The quantum situation is analogous – only the component of  $\vec{\mu}$  parallel to  $\vec{B}$  contributes to the force on the dipole (if the period of the precession is short compared to the

time the field acts on the dipole). Incidentally, this precession of a magnetic dipole moment can result in the phenomenon of *spin magnetic resonance*, which has many modern applications.

Assume that at the location of the dipole the direction of  $\vec{B}$  is along  $\hat{z}$ ; the quantum expression for the force on a single electron becomes (recall that for an electron  $g_e = 2$  and  $m = \pm \frac{1}{2}$ ):

$$\vec{F} = g\mu_B m \frac{\partial B_z}{\partial z} \hat{z} = \pm \mu_B \frac{\partial B_z}{\partial z} \hat{z} \quad (8)$$

If a magnetic dipole moment is due solely to the spin of a single electron, then the magnitude of the force an external magnetic field gradient exerts can take on only one value, and the direction of the force is either parallel or antiparallel to the field at the position of the dipole. A classical magnetic dipole experiences an instantaneous force given by equation (6), where the dot product could take on any value between  $\pm |\vec{\mu}| |\vec{B}|$ . If the duration of the interaction is long, however, the precession of the dipole (figure 1) would again lead to an average force either parallel or antiparallel to the field at the position of the dipole. Unlike the quantum system, its magnitude could vary continuously depending on the angle between the dipole and the field.

## THE STERN-GERLACH APPARATUS

**General arrangement.** Figure 3 (next page) shows a sketch of the arrangement of the oven, atomic beam, magnet, and detector. Potassium metal in the oven is heated to approximately 116 °C. The oven is heated by a current of approximately ½ Ampere; an electronic feedback circuit controls the heating current to maintain a constant oven temperature, as measured by a thermocouple. Atoms evaporated from the potassium in the oven stream through the slit  $S_1$  and are collimated into a narrow, vertical beam by slits  $S_2$  and  $S_3$ . The beam enters a region between the two magnet pole pieces which has a non-uniform magnetic field as described in this experiment's Appendix A. The field imparts a horizontal velocity to the various potassium atoms depending on the orientations of their individual magnetic dipole moments and their vertical velocities. The atoms leave the field and coast upwards to the detection region. The detector can be moved horizontally using a micrometer; it produces a small current which is designed to be proportional to the flux of potassium atoms at its position. This experiment's Appendix B derives the expected distribution of atom flux as a function of detector position.

**The detector.** The detector is a 0.002 inch diameter tungsten wire heated to approximately 1100 °K by a 300 mA current. Surrounding the wire is a cool collection conductor at a potential of approximately 8 Volts below that of the wire. When a potassium atom hits the wire, it is energetically favorable for its valence electron to transfer to the tungsten. The resulting positive potassium ion boils off the hot wire and is attracted to the surrounding collection conductor, where it picks up an electron from the metal and becomes a neutral atom. The flow of ions from

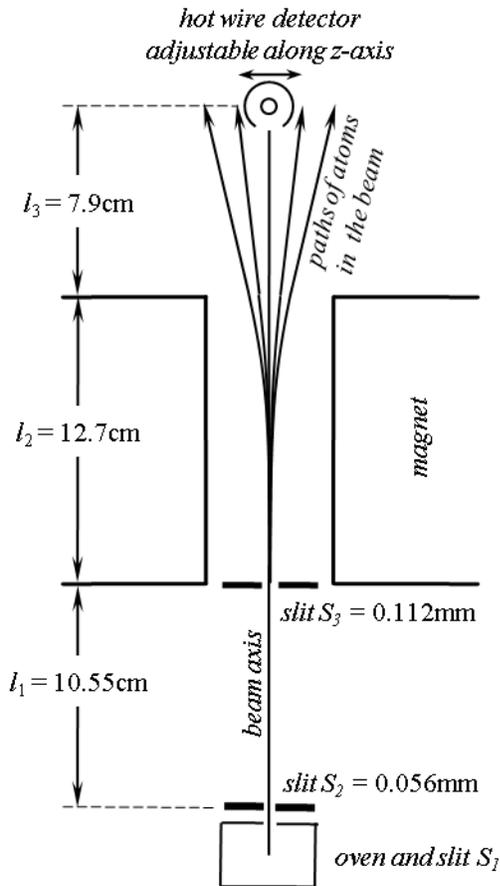


Figure 3: The heart of the apparatus, with the oven, atomic beam, magnet, and detector.

the wire to the collector creates a current (on the order of a few picoamperes or less) which is measured by a sensitive ammeter.

**Vacuum system.** In order for the potassium atoms to have a reasonable chance of reaching the detector without being scattered by some offending air molecule, the system must have a fairly good vacuum (see the prelab problems). An oil diffusion pump followed by a rotating vane pump provides this vacuum. These pumps should remain on throughout the term, even when the apparatus is not being used. Otherwise, the potassium metal in the oven will quickly oxidize and the system will stop working. The vacuum is measured using an ion pressure gauge. Don't activate the gauge while taking measurements of the potassium beam, because it increases the background current in the hot wire detector. The notes for Physics 6, experiment 9 describe the vacuum system in more detail.

**Magnet power supply.** The magnetic field is generated by a large electromagnet powered by an adjustable, regulated power supply. Because the field is determined by the magnet current, the power supply should be used in its constant-current mode. There is an accurate ammeter inserted in the circuit, along with a knife-switch which can reverse the current through the magnet.

Never switch off the power supply or reverse the current flow in the magnet unless the power supply current first is reduced to zero!

The magnet power supply can produce dangerously high currents at over 100 Volts (DC, of course), so don't allow yourself to become a part of the magnet circuit!

Because the magnet's pole pieces are ferromagnetic, they are magnetized by the magnetic field. Consequently, reducing the current to zero does not completely eliminate the magnetic field. The reversing switch allows you to add a small current in the opposite direction to cancel this permanent magnetization and set the field to zero. The ferromagnetism also results in *hysteresis*, which makes the relationship between the power supply current and the resulting field dependent on the history of the past current settings. Since the field cannot be directly measured, you must use a chart (Figure 4, next page) which converts current to field strength. This chart is only

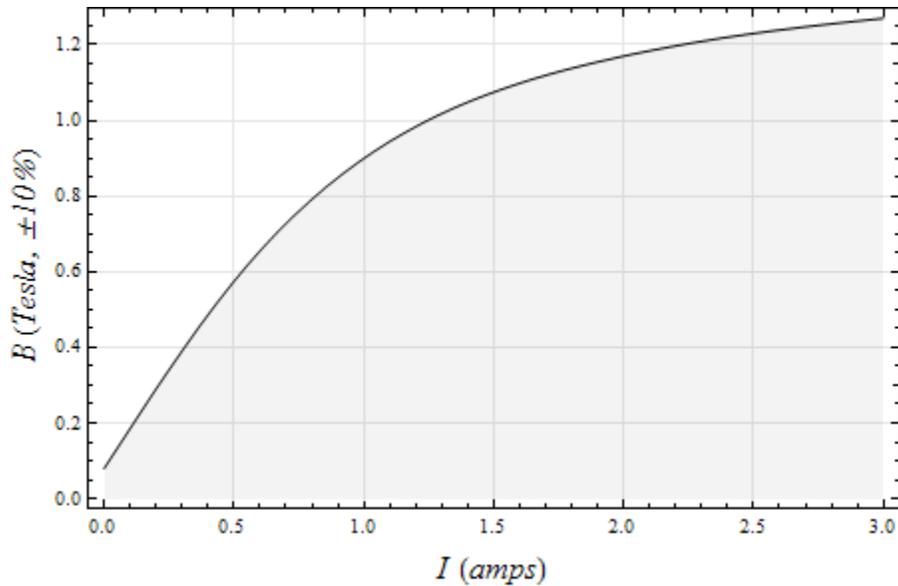


Figure 4: Calibration of the magnet. Because of hysteresis, this curve is only correct for magnet currents decreasing monotonically from saturation (above 3.5 amps). The calibration is  $\pm 10\%$  of the  $B$  value shown. A *Mathematica* notebook is available in the documents folder of the sophomore lab network drive which calculates the field strength given the magnet current.

accurate when the current has been first increased to a maximum, driving the magnet into *saturation*, and then monotonically decreased to the desired value. A *Mathematica* notebook is available on the lab network drive which provides a current to magnetic field calculator using the calibration shown in Figure 4.

Anytime you change the magnet current (even to set it to a lower value), run the current all the way to maximum and then reduce it monotonically to the new value you wish to set. If you inadvertently increase the current, even slightly, you must run the current all the way to maximum and then reset it as described. Otherwise hysteresis makes the chart showing the magnetic field value inaccurate.

## DEFLECTION OF ATOMS IN THE BEAM

The magnet pole pieces are shaped so that the magnetic field has a nonzero gradient which is not particularly difficult to calculate. A description of the pole piece design and a calculation of the field gradient are given in this experiment's Appendix A. The beam of neutral potassium atoms from the oven has a continuous distribution of vertical velocities because of its thermal origin. This velocity distribution is derived in this experiment's Appendix B. This appendix also derives the resultant distribution of atom deflections due to the field gradient's force on their individual magnetic moments. We refer to this continuous distribution of deflections as the *Z Distribution*. An example of the distribution is shown in Figure 5 (next page). Note how the finite width of the atomic beam from the oven can distort the observed distribution. This effect may be

approximated by performing a *convolution* of the ideal distribution with the shape of the 0-field atomic beam. The two peaks of the Z Distribution correspond to the two lines observed by Stern and Gerlach (right hand image in Figure 1).

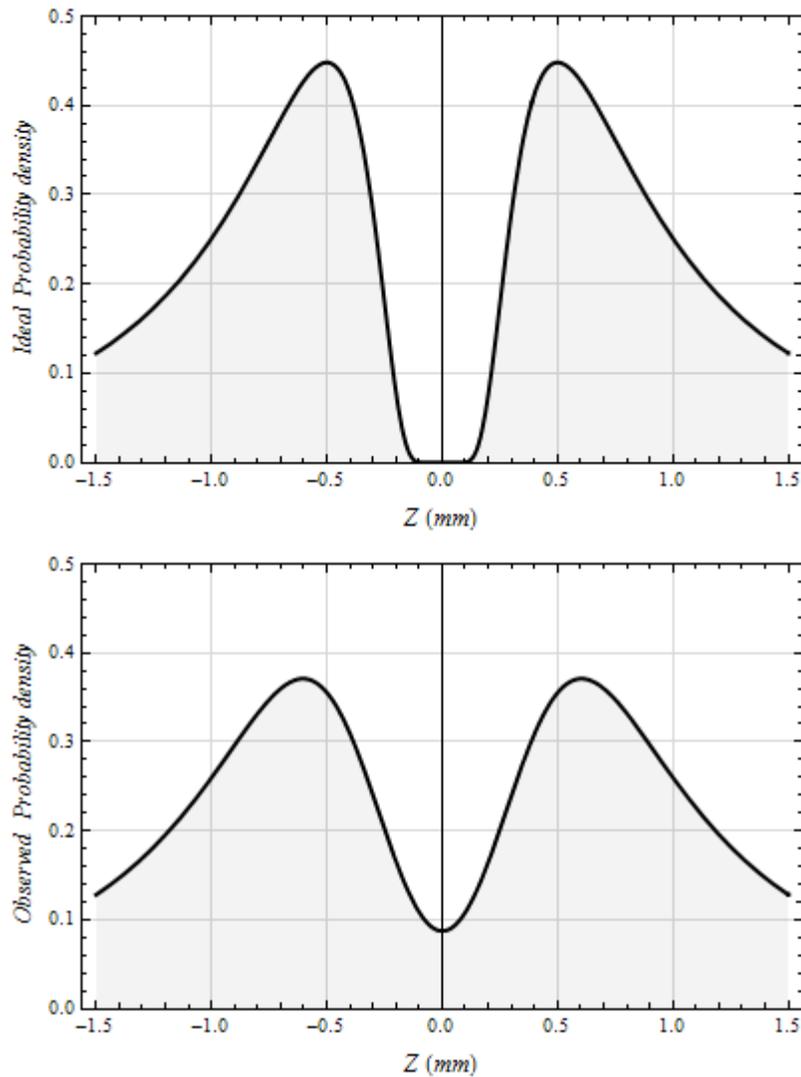


Figure 5: Plots of the *Z Distribution* for an oven temperature of 116 °C and a magnetic field of 0.546 Tesla. The top plot shows the ideal case of a perfectly-collimated atomic beam. The bottom plot shows a more realistic case of a Gaussian beam with a standard deviation of 0.2 mm. The maximum intensity occurs at deflections of  $\pm 0.5$  mm for the ideal case; the finite beam width increases this to  $\pm 0.6$  mm (bottom plot). The ideal *Z Distribution* is derived in this experiment's Appendix B. A *Mathematica* notebook is available in the documents folder of the sophomore lab network drive which calculates profiles like those plotted here.

## PROCEDURE

1. Check that the vacuum is no more than a few  $\times 10^{-5}$  Torr. If it is too high, seek assistance from your TA and the lab administrator. Do not leave the ion gauge activated while performing measurements of the atomic beam detector current.
2. Turn on the power supplies for the oven and the hot wire detector. The warm-up current for the oven should be just over  $\frac{1}{2}$  Ampere; the hot wire current should be approximately 300 mA. The detector current when the oven is cool should be no more than  $10^{-13}$  Ampere. Adjust the micrometer to the approximate beam position, which is between 15.0 and 15.5 mm on the micrometer. Ensure the magnet power supply current is set to 0 and that the magnet power supply is on.
3. The beam should be detectable when the oven temperature reaches  $100^\circ\text{C}$ . Adjust the micrometer to find the center of the beam, and then slightly adjust the magnet current to maximize the detector current at the beam position (if necessary, use the knife switch to reverse the current direction). This ensures that the magnetic field is 0, even with the magnet's hysteresis. Do not take data until the oven temperature has stabilized.
4. Increase the magnet current to about 1 Ampere and note that the detector current decreases. Run the magnet to saturation and then reduce the current to about .5 – 1.0 Ampere. Locate the two peaks in the detector current, one on either side of the 0-field beam position. Note how long it takes the detector current to stabilize when you change the detector position. Record the peak positions. You can estimate uncertainties in a peak position by moving off the peak and finding it again. By repeating this a few times, you can use the distribution of measurements to obtain a peak position and uncertainty.
5. Check your peak separation against the predictions of the theory for your magnetic field and oven temperature. If the difference is more than 20% – 30%, check that you understand what you are doing with your TA. The theory predicts that for an oven temperature of  $116^\circ\text{C}$ , the positions of the peaks are given by (see Appendices A and B):

$$z_{\max} = 0.916 \text{ mm} \times B / \text{Tesla} \quad (\text{oven temperature } 116^\circ\text{C})$$

6. Find the pairs of peaks for several magnet current settings. A large field makes the peaks low and broad, so getting accurate positions in this case is difficult. Make sure you are comfortable with the instrument before attempting the hard data points.
7. Obtain a detector current v. position profile for at least one magnetic field near 0.8 Tesla. Obtain a 0-field profile as well (review step 3 above and make sure the field is really 0).

When finished, make sure the magnet current is zero and the oven and hot wire power supplies are secured. **Leave the vacuum pumps on!**

## ANALYSIS

The point of the analysis is to test the quantum theory as expressed by equation (8), so you must address the following questions:

- (1) How many peaks are observed in the detector current as a function of detector position ( $z$ )?
- (2) Are the positions of the peaks in accord with the theory, as augmented by the magnetic field gradient analysis of Appendix A and the velocity distribution in the atomic beam as analyzed in Appendix B? The separation of the peaks for the ideal Z Distribution is  $\Delta Z_{\text{peaks}} = 2 z_{\text{max}}$ , with  $z_{\text{max}}$  given by (equations (A-18), (B-21), and (B-22)):

$$\frac{\Delta Z_{\text{peaks}}}{2} = z_{\text{max}} = \frac{\beta}{3}$$

$$\beta = \mu_B \left[ l_2 \left( \frac{l_2}{2} + l_3 \right) \frac{1}{2kT} \right] \frac{\partial B_z}{\partial z}$$

$$\frac{\partial B_z}{\partial z} = \frac{1.76}{\text{cm}} B_z$$

Note that  $z_{\text{max}}$  doesn't depend on the mass of a potassium atom! The important idea from the equations above is that  $z_{\text{max}}$  should be linear in the field  $B$ . To what level of accuracy are your results consistent with this claim?

To accurately determine  $z_{\text{max}}$  from your peak position data, you should correct for the effect of the finite 0-field beam width. Use your 0-field data to estimate this width, and then use the *Mathematica* notebook *Z Distribution Calculator.nb* to correct your observed  $z_{\text{max}}$  to the ideal  $z_{\text{max}}$  for a 0-width beam. Compare your measured nonzero magnetic field beam profile against the theory. You will have to scale the theoretical prediction and add a linear background to it in order to make a valid comparison with your data.

Assuming that your results are consistent with the theory, what is your estimate of the value of the Bohr Magneton,  $\mu_B$ ? What is the uncertainty in your estimate? How have you accounted for the magnet calibration uncertainty of  $\pm 10\%$ ?

## REFERENCES

1. Friedrich, B. and Herschbach, D.: Stern and Gerlach: How a bad cigar helped reorient atomic physics, *Physics Today*, December, 2003.
2. Cohen-Tannoudji, C., et al.: *Quantum Mechanics*, vol I, John Wiley and sons, 1977.

## MATHEMATICA NOTEBOOKS FOR STERN-GERLACH DATA ANALYSIS

The sophomore lab network drive has a folder of *Mathematica* notebooks useful for analyzing this experiment's data. They can be found in the *documents* directory, in a subfolder of the *Mathematica Data Analysis* folder. A ReadMe file gives a brief description of each notebook. Copy the entire subfolder to your computer to use the notebooks.

### PRELAB PROBLEMS

1. Calculate the maximum chamber pressure for a potassium atom in the oven beam to have at least a 0.99 probability of reaching the detector hot wire (assume a 30 cm path length and that both the potassium atoms and the air molecules have diameters of 2 Å).
2. Use the *Mathematica* notebooks *Zmax Calculator.nb* and *Z Distribution Calculator.nb* to determine  $z_{\max}$  and to plot a Z Distribution profile for a magnetic field of 1 Tesla and an oven temperature of 116 °C. Use *Magnet Field Calculator.nb* to determine the required magnet current for this field strength.
3. What is the ionization potential of potassium? What is the value of the Boltzmann factor  $\exp(-E_{ion} / kT)$  for an oven temperature of 116 °C? This incredibly tiny number indicates how unlikely it is for the potassium beam to contain a significant number of ions rather than neutral atoms.
4. Setting  $mv^2 \sim kT$  provides an estimate of the average speed of the atoms in the oven. What is this speed for an oven temperature of 116 °C? For how long are the atoms exposed to the magnetic field (distance =  $l_2$ )?
5. A student asks why electrons are not used in this experiment, rather than neutral potassium atoms (many quantum mechanics textbooks use a so-called "Stern-Gerlach" apparatus to sort electrons by spin polarization to perform various thought experiments). It is, after all, the spin of a single atomic electron that gives rise to the deflection of a potassium atom from the oven. Calculate the ratio of the force on the electron's magnetic dipole moment to the Lorentz force  $\vec{F} = q_e \vec{v} \times \vec{B}$ , where the velocity is given by the answer to the previous problem (use equations (8) and (A-18) to calculate the force on the dipole moment). What are the relative directions of these two forces, assuming  $\vec{v} \perp \vec{B}$ ?

## APPENDIX A

THE MAGNETIC FIELD AND ITS GRADIENT

To produce a magnetic field with a gradient is trivial, the problem is that although it is very easy to produce such a field it is, in general, difficult to calculate or to measure its gradient. What we will do is to invent a field that can be calculated analytically, and then we will invent a practical way to produce such a field.

Consider two infinitely long wires normal to the  $y$ - $z$  plane carrying an electric current  $I$  as shown in Fig. A-1. The current in the wire located at  $y = a$  goes in the positive  $x$  direction (into the plane of the paper), while the current in the wire located at  $y = -a$  flows in the negative  $x$  direction (out of the plane of the paper). The magnetic field at a point  $(y, z)$  is given by

$$\vec{B} = \frac{2\pi I}{4\pi\epsilon_0 c^2} \left( \frac{\hat{e}_x \times \vec{r}_a}{r_a^2} - \frac{\hat{e}_x \times \vec{r}_b}{r_b^2} \right) = \frac{D}{r_a^2 r_b^2} (r_b^2 \hat{e}_x \times \vec{r}_a - r_a^2 \hat{e}_x \times \vec{r}_b) \quad (\text{A-1})$$

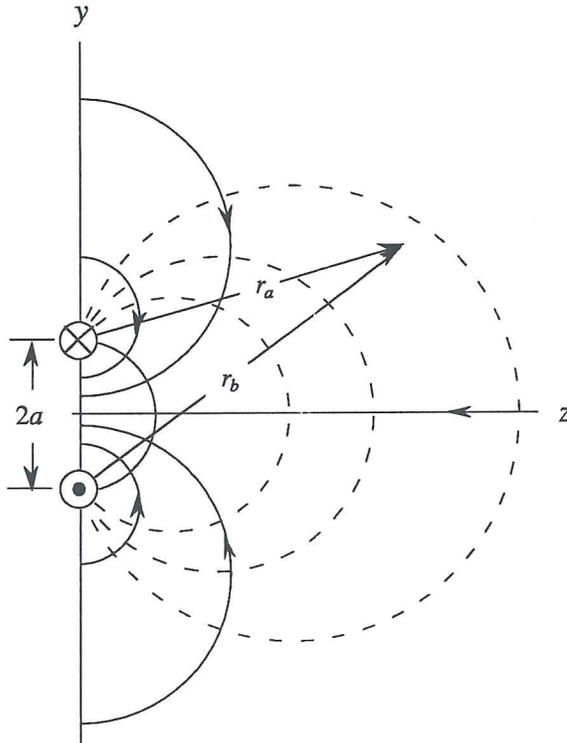


Figure A-1

The magnetic field due to two infinitely long wires.

where  $D = \frac{2\pi I}{4\pi\epsilon_0 c^2}$  and  $\vec{r}_a$  and  $\vec{r}_b$  are defined in Fig. A-1. These two last vectors are given below.

$$\begin{aligned} \vec{r}_a &= z\hat{e}_z + (y-a)\hat{e}_y \\ \vec{r}_b &= z\hat{e}_z + (y+a)\hat{e}_y \end{aligned} \quad (\text{A-2})$$

Inserting Eqs. A-2 in Eq. A-1 we obtain,

$$\begin{aligned} -\frac{r_a^2 r_b^2}{D} \vec{B} &= z(r_b^2 - r_a^2)\hat{e}_y \\ &+ [a(r_b^2 + r_a^2) - y(r_b^2 - r_a^2)]\hat{e}_z \end{aligned} \quad (\text{A-3})$$

The following relations are obvious,

$$\begin{aligned} r_a^2 &= z^2 + (y-a)^2 \\ r_b^2 &= z^2 + (y+a)^2 \end{aligned} \quad (\text{A-4})$$

and from these relations we obtain,

$$r_b^2 - r_a^2 = 4ay \quad (\text{A-5})$$

and

$$r_b^2 + r_a^2 = 2(z^2 + y^2 + a^2) \quad (\text{A-6})$$

Inserting Eqs. A-5 and A-6 in Eq. A-3 we obtain,

$$-\frac{r_a^2 r_b^2}{2aD} \vec{B} = 2yz \hat{e}_y + (z^2 - y^2 + a^2) \hat{e}_z \quad (\text{A-7})$$

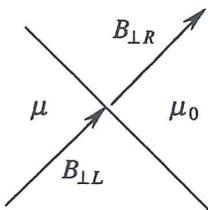
From Eq. A-7 you can find (with some hard work) the following remarkable facts about this field. The lines of the  $B$  field follow circles whose centers are located along the  $y$  axis, as shown by the solid lines in Fig. A-1, and the magnetic field is normal to the surfaces of cylinders whose centers are located along the  $z$  axis, as shown by the dashed circles in Fig. A-1. In this experiment the potassium beam is centered around  $y = 0$  so the magnetic field is given by

$$B_y(y=0) = 0 \quad B_z(y=0) = -2D \frac{a}{a^2 + z^2} \quad (\text{A-8})$$

and taking the gradient of  $B_z$  along the  $z$  direction we obtain,

$$\partial B_z / \partial z = -2 \frac{z}{a^2 + z^2} B_z \quad (\text{A-9})$$

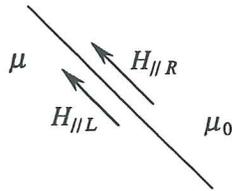
As you will calculate, to obtain a displacement of 1 mm one needs a  $B_z$  field of the order of one Tesla. To obtain this field using two parallel wires a current of the order of  $10^5$  amps (!) is required. This is clearly not a good solution. To produce this field in a practical way we have to use iron core electromagnets.



Let's remember the continuity conditions for the magnetic field at the interface between materials with different permeabilities (see Purcell, Ch 11, pages 436 and 437). Consider the interface between a material with permeability  $\mu$  on the left and one with permeability  $\mu_0$  (vacuum) on the right. First let's consider the components normal to the

the interface. From  $\vec{\nabla} \cdot \vec{B} = 0$

$$B_{\perp L} = B_{\perp R} \quad (\text{A-10})$$



Now let's consider the components parallel to the interface. From

$$\vec{\nabla} \times \vec{H} = 0$$

$$H_{\parallel L} = H_{\parallel R} \quad (\text{A-11})$$

Since

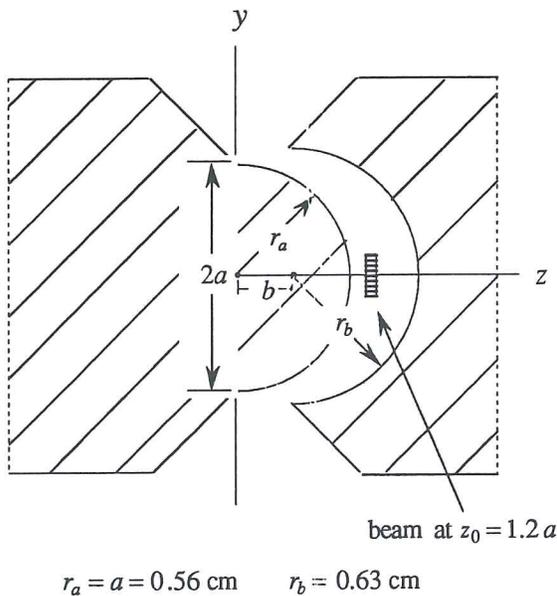
$$\vec{B} = \mu \vec{H} \quad (\text{A-12})$$

From Eqs. A-11 and A-12,

$$B_{\parallel R} = \frac{\mu_0}{\mu} B_{\parallel L} \quad (\text{A-13})$$

For iron  $\frac{\mu}{\mu_0}$  is around  $10^4$ , therefore from Eq. A-13

$$B_{\parallel R} \approx 0 \quad (\text{A-14})$$



**Figure A-2**

The shaped pole pieces of the magnet.

That is, the magnetic field on the outside surface of a high permeability material is normal to its surface (reminds you of the electric field on the surface of an electrical conductor?). The solution used in this experiment is to use an iron electromagnet with its pole pieces shaped as shown in Fig. A-2. The two cylindrical sections of the pole pieces are made so they coincide with two of the surfaces shown by the dashed lines in Fig. A-1, where the magnetic field is normal to these surfaces. In this case the field in the central region corresponds to the field produced by the two wires, and its gradient is given by Eq A-9 (again, see

Purcell Ch 3, page 94. In vacuum a static magnetic field satisfies Laplace's equation and, therefore, it is completely determined by its boundary conditions. You will understand all this stuff when you take Ph 106).

Numbers: in this experiment:  $r_a = a = 0.56$  cm,  $r_b = 0.63$  cm and the beam position is at  $z_0 = 1.2 \times a = 0.67$  cm. The center of the  $b$  cylinder is located at  $z = 0.29$  cm and, therefore, the space between the two cylinders along the  $z$  axis is 0.36 cm.

The relation between the separation  $2a$ , the radius  $r$  and the  $z$  position of the center of the cylinder is given by

$$b^2 = r^2 - a^2 \quad (\text{A-15})$$

Let's redefine the origin of the  $z$  axis at the beam position, so the  $z$  variable in Eq. A-7 is given by

$$z = 1.2 \times a + z \quad (\text{A-16})$$

With this definition of the origin of the  $z$  axis, Eq. A-9 is given by

$$\frac{\partial B_z}{\partial z} = -2 \frac{z + 1.2 \times a}{a^2 + (z + 1.2 \times a)^2} B_z \quad (\text{A-17})$$

Using  $a = 0.56$  cm and  $z = 0$  (at the beam position) we obtain

$$\frac{\partial B_z}{\partial z} = -2 \frac{1.2 \times a}{(1 + 1.2)a^2} B_z = -2 \frac{1.2}{(1 + (1.2)^2)a} B_z = -1.76 B_z / \text{cm} \quad (\text{A-18})$$

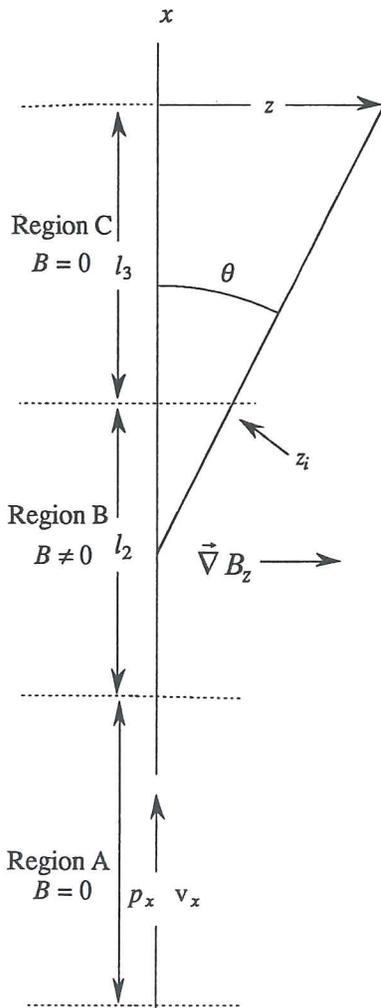
## APPENDIX B

### THE Z DISTRIBUTION

In order to obtain the  $z$  distribution we must calculate the following:

- a- Find the relation between the momentum of the atoms and the  $z$ -position at the detector's position.
- b- Find the momentum distribution of the flux of atoms coming out of the oven.
- c- Transform the  $p_x$  distribution to the  $z$  distribution.

a- Relation between the momentum and the  $z$  position at the detector. In Fig. B-1 we show



**Figure B-1**

the approximate geometry of this experiment. We will neglect the fringe magnetic fields so the magnetic field vanishes in regions A and C. The force on the potassium magnetic dipole is given by Eq. 7

$$F_z = \pm \mu_B \frac{\partial B_z}{\partial z} \quad (\text{B-1})$$

with the gradient given by Eq. A-17

$$\frac{\partial B_z}{\partial z} = -2 \frac{z + 1.2 \times a}{a^2 + (z + 1.2 \times a)^2} B_z \quad (\text{B-2})$$

where  $z = 0$  is the position of the unbent beam and  $a = 0.56$  cm, as given in App. A. Since the beam deflection inside the magnetic field is very small we will assume that  $z$  is constant and equal to 0. The acceleration in the  $z$  direction,  $a_z$ , is obtained from Eqs. B-1 and B-2,

$$a_z = \frac{F_z}{m} = \pm 2 \frac{\mu_B}{m} \frac{1.2}{(1 + 1.2^2)a} B_z = \pm \frac{\mu_B}{m} \times 1.76 B_z \quad (\text{B-3})$$

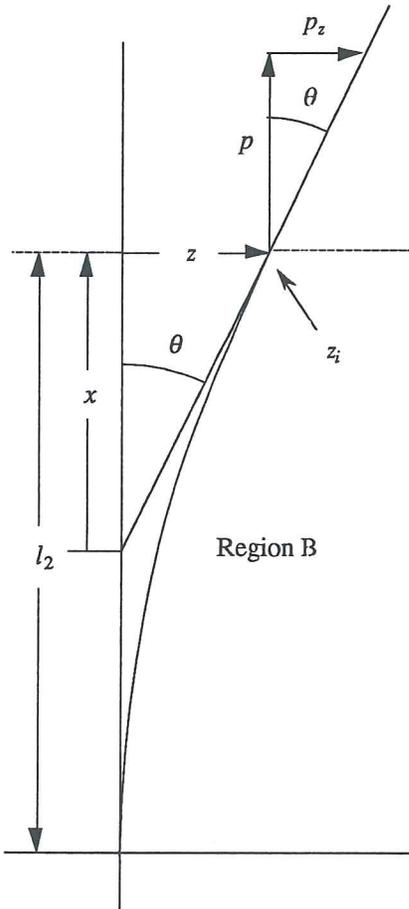
where  $m$  is the mass of the potassium atom. From now on we will drop the  $\pm$  sign. The  $z$  displacement, measured from the unbent beam position, at the exit of

the magnet (region B) is given by

$$z_i = \frac{a_z t^2}{2} \quad (\text{B-4})$$

where  $t$  is the time taken by the atoms to travel through region B. Since there is no force in the  $x$  direction this time is given by

$$t = \frac{l_2}{v_x} = \frac{m l_2}{p_x} \quad (\text{B-5})$$



The displacement in the  $z$  direction at the exit of the magnet (region B) is therefore given by

$$z_i = \frac{m^2 a_z l_2^2}{2 p_x^2} \quad (\text{B-6})$$

Now let's calculate the beam angle at the exit of the magnet. The acquired momentum in the  $z$  direction is given by

$$p_z = m v_z = m a_z t = m a_z \frac{l_2}{v_x} = \frac{m^2 a_z l_2}{p_x} \quad (\text{B-7})$$

Since this angle is very small we can approximate it by

$$\theta \approx \frac{p_z}{p_x} = \frac{m^2 a_z l_2}{p_x^2} \quad (\text{B-8})$$

The total  $z$  displacement is given by

$$z = z_i + \theta l_3 = \frac{m^2 a_z l_2^2}{2 p_x^2} + \theta l_3 \quad (\text{B-9})$$

Figure B-2

Inserting Eq. B-8 in Eq. B-9 we obtain

$$z = l_2 \left( \frac{l_2}{2} + l_3 \right) \frac{m^2 a_z}{p_x^2} \quad (\text{B-10})$$

as shown in Fig. B-2. From Eq. B-10

$$p_x = \sqrt{\frac{\alpha}{z}} \quad (\text{B-11})$$

with

$$\alpha = l_2 \left( \frac{l_2}{2} + l_3 \right) m^2 a_z \quad (\text{B-12})$$

b- The momentum distribution of the flux of atoms coming out of the oven. Since we are only interested in the momentum dependence of the flux of atoms we can use dimensional analysis to obtain this flux. The dimensions of  $dn_v$  (atoms/unit volume- $dv$ ) are  $1/l^3 v$ , and the dimensions of  $d flux$  (atoms/unit area-unit time- $dv$ ) are  $1/l^2 t v$  the ratio of  $d flux/dn_v$  has the dimension of velocity, and since the only speed involved is the atomic speed  $v$ ,

$$d flux \propto v dn_v$$

The exact relation is

$$d flux = \frac{1}{4} v dn_v \quad (\text{B-13})$$

$dn_v$  is given by the Boltzmann distribution of molecular speeds,

$$dn_v \propto v^2 e^{-\frac{mv^2}{2kT}} dv \quad (\text{B-14})$$

Inserting Eq. B-14 in Eq. B-13 we obtain,

$$d flux \propto v^3 e^{-\frac{mv^2}{2kT}} dv$$

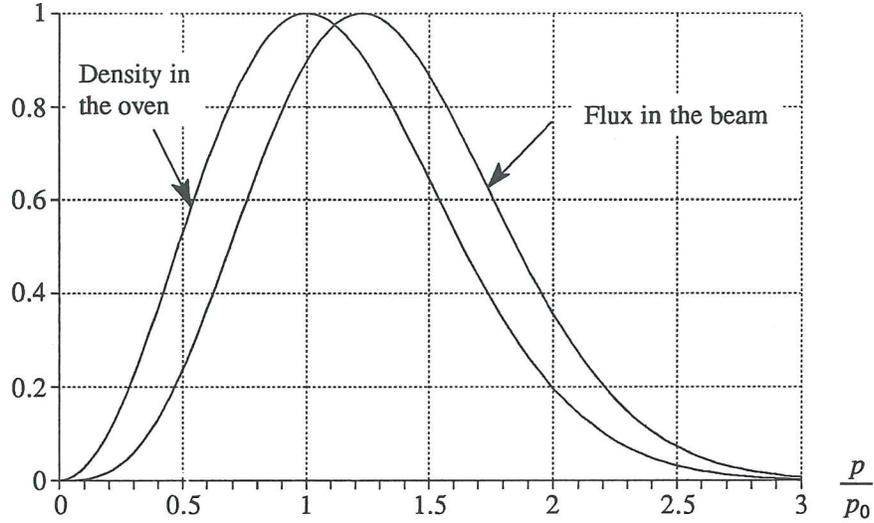
or in terms of the momentum,

$$d flux \propto p^3 e^{-\frac{p^2}{2mkT}} dp \quad (\text{B-15})$$

Therefore the momentum distribution of the atoms leaving the oven is given by,

$$N(p_x) dp_x \propto p_x^3 e^{-\frac{E}{kT}} dp_x = p_x^3 e^{-\frac{p_x^2}{2mkT}} dp_x \quad (\text{B-16})$$

In Fig. B-3 we compare the density distribution given by Eq. B-15 and the flux distribution given by Eq. B-16.



**Figure B-3**

The momentum density and flux distributions.

c- Transform the  $p_x$  distribution to the  $z$  distribution. To transform the  $p_x$  distribution to the  $z$  distribution all we have to do is the following,

$$N(z)dz = N(p_x) \frac{dp_x}{dz} dz \propto p_x^3 e^{-\frac{p_x^2}{2mkT}} \frac{dp_x}{dz} dz \quad (\text{B-17})$$

From Eq. B-11,

$$\frac{dp_x}{dz} = \text{const. } z^{-3/2} \quad (\text{B-18})$$

Inserting Eqs. B-18 in Eq. B-17 we obtain,

$$N(z)dz = \text{const. } \frac{p_x^3}{z^{3/2}} e^{-\frac{p_x^2}{2mkT}} dz \quad (\text{B-19})$$

And expressing  $p_x$  in terms of  $z$  using Eq. B-11

$$N(z) = \text{const. } z^{-3} e^{-\frac{\beta}{z}} \quad (\text{B-20})$$

Where,

$$\beta = \frac{\alpha}{2mkT} = \frac{l_2 \left( \frac{l_2}{2} + l_3 \right) \mu_B}{2kT} \frac{\partial B_z}{\partial z} \quad (\text{B-21})$$

Now let's find the value of  $z$  corresponding to the maximum in the  $z$  distribution. From Eq. B-20,

$$\frac{dN(z)}{dz} = \left( -3z^{-4} + \beta z^{-3} \frac{1}{z^2} \right) e^{-\frac{\beta}{z}} = 0$$

or

$$\frac{dN(z)}{dz} = z^{-5}(-3z + \beta) = 0$$

therefore

$$z_{\max} = \frac{\beta}{3} \quad (\text{B-22})$$

Let's insert the value of  $\beta$  given by Eq. B-22 in Eq. B-20,

$$N(z) = \text{const. } z^{-3} e^{-3 \frac{z_{\max}}{z}} \quad (\text{B-23})$$

Let's normalize the  $N(z)$  distribution given by Eq. 23 to 1 at  $z = z_{\max}$ . At  $z = z_{\max}$  Eq. B-23 gives,

$$N(z_{\max}) = \text{const. } z_{\max}^{-3} e^{-3} = 1 \quad (\text{B-24})$$

therefore

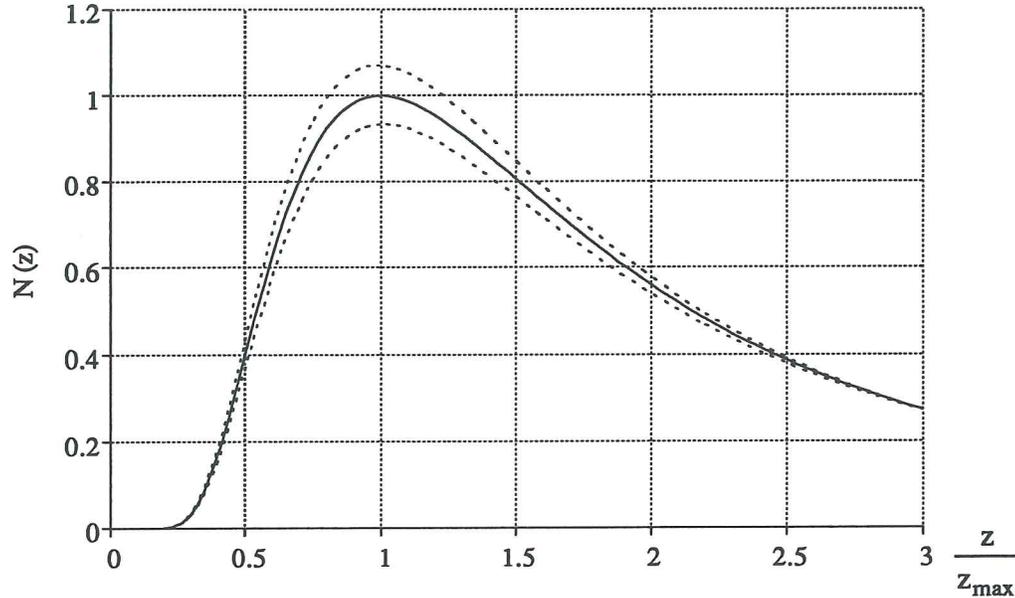
$$\text{const.} = z_{\max}^3 e^3 \quad (\text{B-25})$$

Inserting this value of the const. in Eq. 23 we obtain,

$$N(z) = \left( \frac{z_{\max}}{z} \right)^3 e^{3 \left( 1 - \frac{z_{\max}}{z} \right)} \quad (\text{B-26})$$

This distribution is shown by the solid line in Fig. B-4. The dotted lines give the exact distributions when the gradient of the magnetic field is given by Eq. B-2. Since the

gradient given by this equation increases with  $z$  the bending of the beam for  $z > 0$  is larger than for  $z < 0$  for this reason the distributions are no longer symmetric.



**Figure B-4**

The solid line gives the  $z$  distribution for a constant gradient, the dotted lines for the gradient given by Eq. B-2.

### Numerical Values

Let's calculate  $z_{\max}$ .

$$l_2 = 12.7 \text{ cm} \quad \frac{l_2}{2} + l_3 = 14.25 \text{ cm} \quad \frac{1}{B_z} \frac{\partial B_z}{\partial z} = 1.76 \text{ cm}^{-1}$$

and

$$k = \frac{1}{11604} \text{ eV}/^\circ\text{K}$$

(Notice: 11604 °K is the temperature corresponding to 1 eV.). From Eqs. B-2 (with  $a = 0.56 \text{ cm}$ ), B-21 and B-22 we obtain,

$$z_{\max} = \frac{l_2 \left( \frac{l_2}{2} + l_3 \right) \mu_B}{6kT} \times 1.76 B_z / \text{cm} \quad (\text{B-27})$$

With the numerical values given above and for an oven temperature at 110 °C we obtain from Eq. B-24,

$$z_{\max} (\text{mm}) = 0.931 \times B(\text{Tesla}) \quad \text{at } 110^\circ\text{C}$$