Experiment 12

ELECTRON DIFFRACTION

INTRODUCTION

In this experiment you will investigate the wave-like nature of particles by diffracting electrons as they pass through crystals of aluminum or graphite. Since the structures of these crystals have been accurately determined from the results of X-ray diffraction studies, your results may be used to measure the wavelength of a traveling electron as a function of its momentum. A fundamental relation between momentum and wavelength is predicted by the quantum theory; your analysis will test this theory and determine an accurate value for Planck's constant h.

Nearly twenty years after the discovery of the particle properties of electromagnetic radiation in 1905, the French physicist Louis de Broglie speculated that particles of matter may exhibit wave-like properties as well. Interestingly, his speculation was well received even though there was little if any experimental evidence to support it. His assertions in 1924 were profoundly important to subsequent development of the quantum theory — Erwin Schrödinger, Werner Heisenberg, and others expanded de Broglie's notion and provided the first formal mathematical treatment of quantum mechanics during 1925–1926. In 1927 Davisson and Germer in the U.S. and in 1928 G. P. Thompson in England independently performed experiments demonstrating electron diffraction which confirmed de Broglie's hypothesis.

Since the momentum of a photon is related to its kinetic energy by $pc = E = h\nu$, the momentum and wavelength are related by

$$p = h/\lambda \quad ; \quad \vec{p} = \hbar\vec{k} \tag{1}$$

where $\hbar \equiv h/2\pi$ and $\vec{k}$ is the wave vector of the electromagnetic wave. De Broglie hypothesized that the relation (1) would obtain for any particle, not just the photon. For a non-relativistic particle the kinetic energy $E = p^2/2m$, so

$$k^2 = \frac{2mc^2}{(\hbar c)^2}E \tag{2}$$

DIFFRACTION BY GRAPHITE

The electron beam apparatus used in this experiment has a target consisting of very thin crystals of "pyrolytic" graphite, a pure form of carbon. A graphite crystal consists of planar layers of carbon atoms in which each layer is a honeycomb of interconnected benzene-ring hexagons of carbon atoms (figure 1). The graphite target crystals have their honeycomb layers normal to the electron beam axis and are only several 10's of layers thick. The centers of the individual hexagons within a layer are arranged in a pattern of identical equilateral triangles which tile the layer plane; the length of the side of each triangle is 2.4612 Å (the "lattice constant").

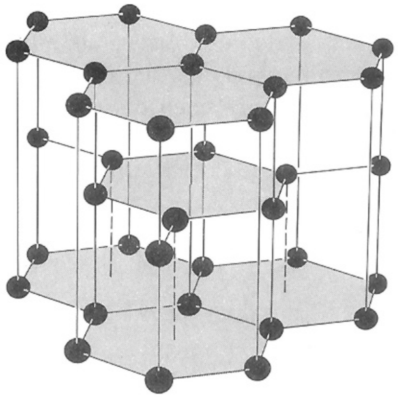


Figure 1: Graphite crystal structure. The crystal consists of layers of benzene-rings of carbon atoms arranged as shown. The distance between centers of adjacent hexagons in a layer is 2.4612 Å, and the layer separation is 3.3539 Å, much larger than the separation of nearest-neighbor carbon atoms within a layer (approximately 1.4 Å; the ratio of vertical and horizontal lengths in the figure is not to scale). This figure is from Reference 1.

To determine the diffraction pattern produced by waves traveling through the very thin graphite, we consider the diffraction from a single layer; the diffraction pattern is changed only slightly by the presence of a few additional layers. A derivation of the diffraction pattern from a graphite layer is developed in Appendix A to this write-up; the results are summarized here.

The "simple hexagonal" planar array of graphite consists of a homogeneous lattice of pairs of carbon atoms (homogeneous in the sense that each pair is surrounded by the same configuration of neighbors, including orientation — a *Bravais lattice*); as a result the position of each lattice site may be expressed as a linear combination of two fundamental basis vectors with integer coefficients:

$$\begin{aligned} &\vec{R}_{ij} = i\,\vec{R}_+ + j\,\vec{R}_- \quad ; \quad i,j \in \mathbb{Z} \\ &\vec{R}_+ = \tfrac{1}{2}a_0\left(\sqrt{3}\,\hat{y} + \hat{x}\right); \vec{R}_- = \tfrac{1}{2}a_0\left(\sqrt{3}\,\hat{y} - \hat{x}\right) \\ &a_0 = 0.24612\,\text{nm for graphite} \end{aligned} \tag{3}$$

As mentioned in Appendix A, the diffracted waves produced by a Bravais lattice have wave vectors whose difference with the incoming wave vector must be a member of the *reciprocal lattice*, a Bravais lattice of wave vectors. This wave vector lattice consists of those plane waves which would have the same phase at every crystal lattice site. In the case of a lattice confined to a single plane such as a single layer of graphite, we project the incoming and outgoing wave vectors onto the plane; the difference in the projected vectors must be a member of the reciprocal lattice. The reciprocal lattice of a simple hexagonal array is also simple hexagonal, but rotated 90° with respect to the original lattice (compare equations (3) and (4)).

$$\begin{aligned}
&\vec{K}_{lm} = l\,\vec{K}_+ + m\,\vec{K}_- \quad ; \quad l,m \in \mathbb{Z} \\
&\vec{K}_+ = \tfrac{1}{2}K_0\left(\sqrt{3}\,\hat{x} + \hat{y}\right);\, \vec{K}_- = \tfrac{1}{2}K_0\left(\sqrt{3}\,\hat{x} - \hat{y}\right) \\
&K_0 = 4\pi / \left(\sqrt{3}\,a_0\right)
\end{aligned} \tag{4}$$

The diffraction pattern from a single, thin crystal of graphite due to an incoming beam normal to the crystal plane is therefore a set of waves with wave vectors whose *x*- and *y*-components are given by equations (4). Since the orientation of a crystal within the *x*-*y* plane is arbitrary, this hexagonal pattern may be rotated about the *z*-axis (the beam axis) by any random angle. If the electron beam illuminates several crystals with different rotational orientations, then the diffracted waves will produce nearly complete rings about the original beam, the various ring radii (in *k*-space) being given by the lengths of the various reciprocal lattice vectors (4).

DIFFRACTION BY ALUMINUM

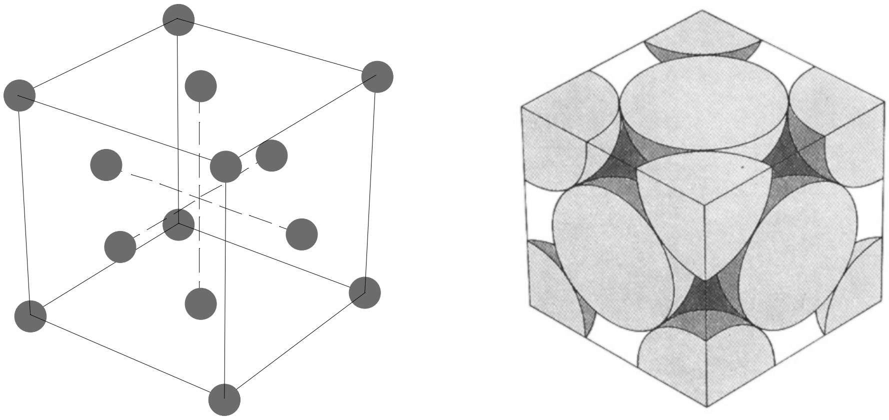


Figure 2: Aluminum crystal structure. The *face-centered cubic* crystal has a cubic unit cell containing 4 atoms arranged as shown. The atoms are arranged as close-packed spheres in 3-dimensional space similar to a tetrahedral pyramid of cannonballs, as can be seen by examining the relative positions of the three atoms in the centers of the top, left, and right faces. The length of a side of the unit cell is the lattice constant and is 4.050 Å. Each atom has 12 nearest neighbors separated from it by 2.86 Å. The graphic on the right is from Reference 1.

The aluminum targets, although thin, are polycrystalline with the orientation of each crystal being random in 3-dimensional space. Since the electron beam will illuminate many of these randomly-oriented crystals, the diffraction pattern consists of rings rather than the discrete dots from particular reciprocal lattice vectors with a definite orientation. The arrangement of aluminum atoms within a crystal is called *face-centered cubic* (or *fcc*) and is one of the most common crystal structures. The conventional Bravais lattice for the *fcc* crystal is a cubic array of copies of the unit cell shown in figure 2. Although not readily apparent from the figure, the *fcc* structure is a close-packed array of spheres identical to a tetrahedral pyramid of stacked cannonballs. The lattice constant a_0 for the Al crystal is the length of one side of the *fcc* unit cell and is equal to 4.050 Å.

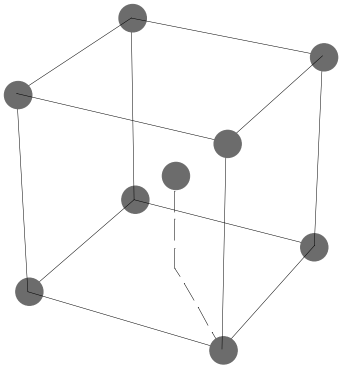
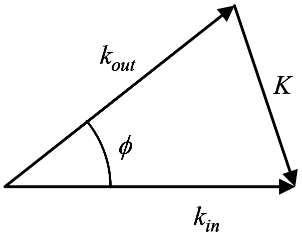


Figure 3: a conventional unit cell of the *body-centered cubic* structure of the reciprocal lattice of aluminum's *face-centered cubic* crystal shown in figure 2. Each point represents the components of one of the reciprocal lattice wave vectors; of course, the cubic cell shown is repeated in all directions to completely fill $\vec{K}$ space. The *bcc* points are not in a close-packed arrangement as was the *fcc* structure. The lattice constant (the length of the side of the unit cell) is $K_0 = 4\pi / a_0$; the length of the shortest $\vec{K}$ vector is equal to the distance from a corner to the center lattice point of the cube, which is $\sqrt{3}K_0 / 2$.

The reciprocal lattice for aluminum's *fcc* arrangement is called *body-centered cubic* (or *bcc*) and is shown in figure 3. The lattice constant for the reciprocal lattice (the length of the side of the *bcc* unit cell) is $4\pi / a_0$ with $a_0 = 4.050\,\text{Å}$. As noted in the previous section, the difference between the outgoing and incoming wave vectors must be a member of the set of reciprocal lattice vectors for constructive interference to occur and a diffracted beam to be observed. Since the scattering is elastic, the incoming and outgoing wave vectors have the same magnitude, so:



$$\sin \tfrac{1}{2}\phi = K / 2k \qquad (5)$$

Where ϕ is the angle between the incoming electron beam and the diffracted beam, K is the magnitude of a reciprocal lattice vector, and k is the wave number of the incoming beam. For a polycrystalline sample and any particular choice of the electron beam k, this relation will be satisfied for many of the various $\vec{K}$ lattice vectors, and the diffracted rays form cones around the direction of the incoming beam with various angles ϕ.

THE ELECTRON DIFFRACTION APPARATUS

The electron beam apparatus consists of a cathode ray tube (CRT) and its associated power supply/control console. The CRT is based on the type of tube used in a conventional analog oscilloscope, modified to contain the thin crystalline target for the electron beam (figure 4). The electron gun generates the electron beam by accelerating and focusing the electrons using a high-voltage power supply. The electrons' kinetic energy may be set to as high as 10keV (kilo-electron-volts); the range for effective operation of the experiment is approximately 5–9 kilovolts. The beam current (intensity) is also controllable; this current should never exceed 10 microamps to minimize permanent damage to the target. Steering electrodes are provided to electrostatically deflect the beam horizontally and vertically, so that various positions on the target may be illuminated by the electron beam.

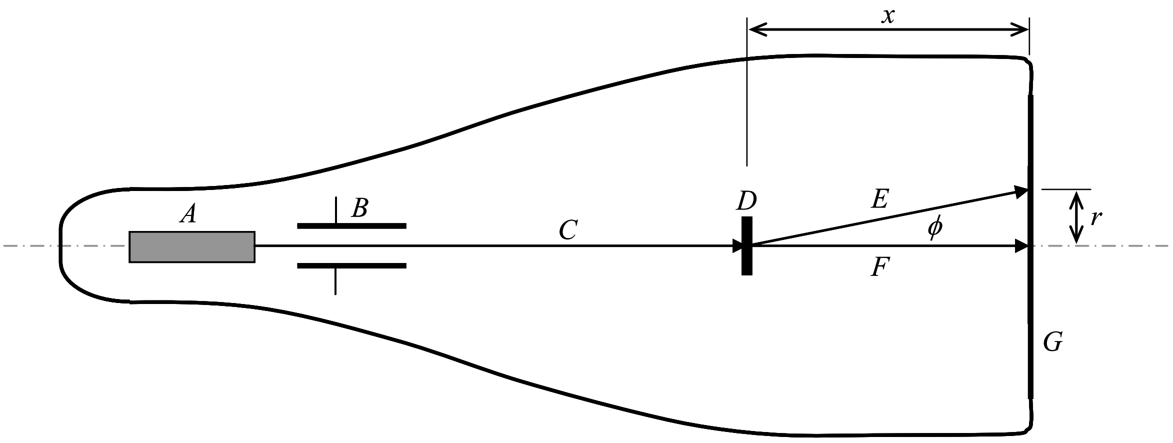


Figure 4: a schematic diagram of the electron beam tube. *A*: electron gun; *B*: beam steering electrodes; *C*: electron beam to target; *D*: target; *E*: diffracted outgoing beam; *F*: direct (undiffracted) outgoing beam; ϕ: angle between direct and diffracted beams; *G*: phosphorescent screen; *x*: distance from target to screen; *r*: screen distance between direct and diffracted beams. The distance *x* is approximately 17.8 cm.

The target is divided into 4 quadrants: looking at the target from in front of the tube, the upper left quadrant contains the thin graphite crystals, and the other three quadrants contain polycrystalline aluminum of varying thicknesses. The beams from the target finally hit a phosphorescent screen (*G* in figure 4) which glows in response to the electron impacts. The screen is flat and covered with a centimeter square grid. The positions of the direct and diffracted beams may be measured on the screen with sufficient accuracy using a plastic ruler. Typical screen images generated during the experiment are shown schematically in figure 5.

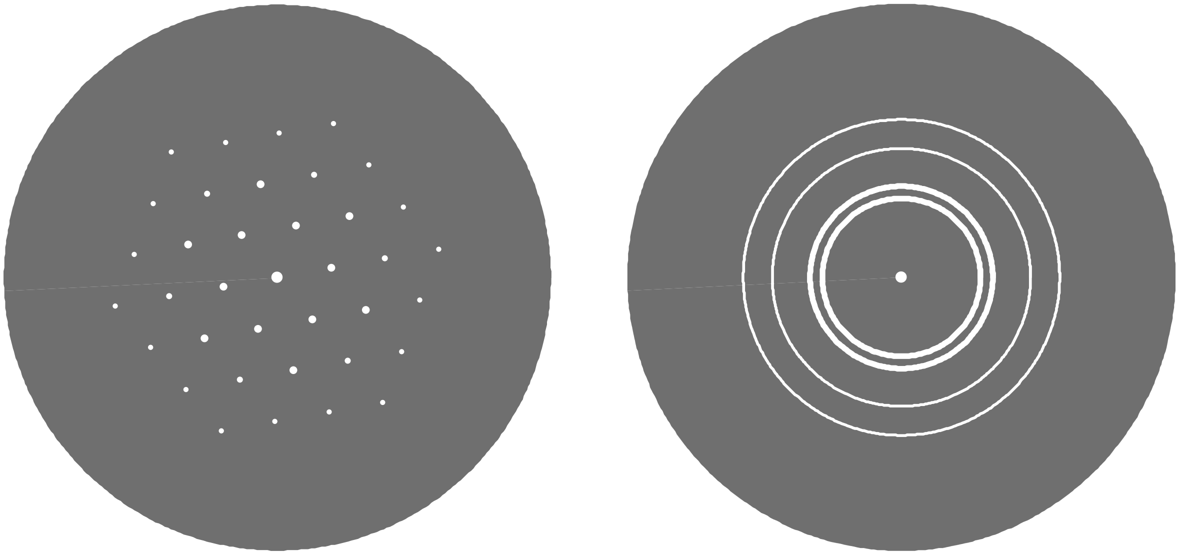


Figure 5: schematic representations of typical diffraction images observed during the experiment. The center dot in each graphic is the undiffracted, or direct, beam. Left: diffraction pattern from a single graphite crystal. Right: first 4 diffraction rings from polycrystalline aluminum.

PROCEDURE AND ANALYSIS

NEVER EXCEED **10 MICROAMPS** TARGET CURRENT!

Higher acceleration voltages require quite small currents for a useable image (1–2 μA). Keep the current at its minimum useable value; turn the current down when you are not taking measurements. Many of the high energy electrons in the beam collide with valence electrons in the target and knock them out of their atoms, which can change bonds and introduce defects into the crystal structure. Eventually high beam currents will destroy the crystal structure and eliminate the coherent diffraction patterns we can now observe.

HIGH VOLTAGE SHOULD BE SET BETWEEN **5 AND 9 KILOVOLTS**!

Start the experiment with a voltage nearer the high end of this range, because the image will be brighter, and points on the target that provide a good diffraction image will be easier to find.

DO NOT USE A SHARP METAL OBJECT TO MEASURE THE DIFFRACTION PATTERN! **DO NOT USE CALIPERS**!

Your data will consist of the sizes of the various diffraction features as a function of accelerating voltage. Multiple measurements using a clear plastic ruler can give you quite accurate data once you average the results; this will also provide an uncertainty determination. Calipers will scratch the plastic plate covering the screen.

Refer to figure 4. If the angle ϕ to a diffraction feature is small, then we can adequately approximate ϕ, $\sin(\phi)$, and $\tan(\phi)$ as r/x. For any particular diffraction feature, the reciprocal lattice vector magnitude K is a constant, so the angle ϕ would then be inversely proportional to the electron beam wave number k (see equation (5)). De Broglie's theory (equation (2)) requires that $k^2 \propto E$ (the kinetic energy), so the relation between r and $V^{-1/2}$ (V is the acceleration voltage) should be linear (see also equations (6) and (7) in the prelab problems on page 12-8). Use your data to test this assertion. Using $mc^2 = 0.511$ MeV for the electron, determine a value (with uncertainty) for Planck's constant h in eV×sec (or, equivalently, the product hc in eV×Å).

Are the values you calculate for h by analyzing your data for the various different diffraction features consistent with each other? If not, what may this imply about possible systematic errors in your measurements? Note from equation (7) that scaling the data from various diffraction features by their $\widehat{K}/2\pi a_0$ values should collapse all data points onto a single line.

REFERENCES

(References are available for review in the lab)

1. Ashcroft, N.W. and Mermin, N.D.: Solid State Physics, Thomson Learning, Inc., 1976.

2. Leighton, R.B.: Principles of Modern Physics, McGraw-Hill, New York, 1959.

3. Beiser, A.: Perspectives of Modern Physics, McGraw-Hill, New York, 1969.

PRELAB PROBLEMS

1. Show that the de Broglie wavelength λ of an electron accelerated through a potential V by the electron gun is:

$$\lambda = \frac{hc}{\sqrt{2mc^2 eV}} \tag{6}$$

Given $hc = 12{,}400\,\text{eVÅ}$ and $mc^2 = 0.511\,\text{MeV}$, what is λ if $V = 8.0\text{kV}$?

2. Define a *normalized* reciprocal lattice vector magnitude $\widehat{K} \equiv a_0 \left|\vec{K}\right|$ of the reciprocal lattice vector $\vec{K}$. The value of $\widehat{K}$ (which is a pure number) depends on the crystal geometry and a particular reciprocal lattice vector $\vec{K}$, but not the size of the crystal, which is characterized by a_0. Show that the normalized reciprocal lattice vector magnitudes $\widehat{K}$ for the first three diffraction circles in aluminum (see figure 5) are $2\pi\sqrt{3}$, 4π, and $4\pi\sqrt{2}$. Use the reciprocal lattice's *bcc* unit cell (figure 3) to guide you.

3. Use equations (4) to show that the normalized reciprocal lattice vector magnitude $\widehat{K}$ (see problem 2) for the 6 diffraction spots closest to the direct beam in graphite (see figure 5) is $4\pi/\sqrt{3}$, and that for the next 6 closest spots $\widehat{K}$ is 4π.

4. Show that:

$$\frac{r}{x} = \frac{\widehat{K}}{2\pi}\,\frac{\lambda}{a_0} \tag{7}$$

Where r and x are defined in figure 4, and we use the small angle approximation $\phi = \sin\phi = \tan\phi = r/x$; a_0 is the lattice constant; λ is the electron de Broglie wavelength from equation (6); and $\widehat{K}$ is the normalized reciprocal lattice vector magnitude for the diffraction pattern feature at r. What should r be for the 6 closest diffraction spots of graphite if $V = 8.0\text{kV}$ (see figure 5)? What should r be for the first diffraction circle of aluminum?

5. The expression (2) is inaccurate for relativistic velocities. The relativistic expressions for the wave parameters k and ω of a traveling particle with rest mass m are:

$$\hbar\omega = E = \gamma mc^2 \quad ; \quad \hbar k = p = \gamma mv \tag{8}$$

Where γ is the "boost" $\left(1 - v^2/c^2\right)^{-1/2}$, E is the total energy (including the rest energy mc^2), and v is the magnitude of the 3-velocity of the particle. Show that:

(a) The phase velocity of the wave described in (8) is always greater than the speed of light.

(b) The group velocity $d\omega/dk = v$, the particle's velocity (and is therefore $< c$). Hint:

$$\frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

APPENDIX A: DIFFRACTION BY A GRAPHITE PLANE

The honeycomb of benzene rings which makes up a layer of graphite is not a *Bravais* lattice if each atom is considered to be a lattice point, because each atom does not see the same configuration of neighbors (including orientation). We can define a *Bravais* lattice representation, however, by defining the lattice points to be the locations of similarly-oriented *pairs* of atoms as in figure 6. The equilateral triangles which comprise this lattice have sides of length $a_0 = 2.4612\,\text{Å}$, the *lattice constant* for a graphite plane.

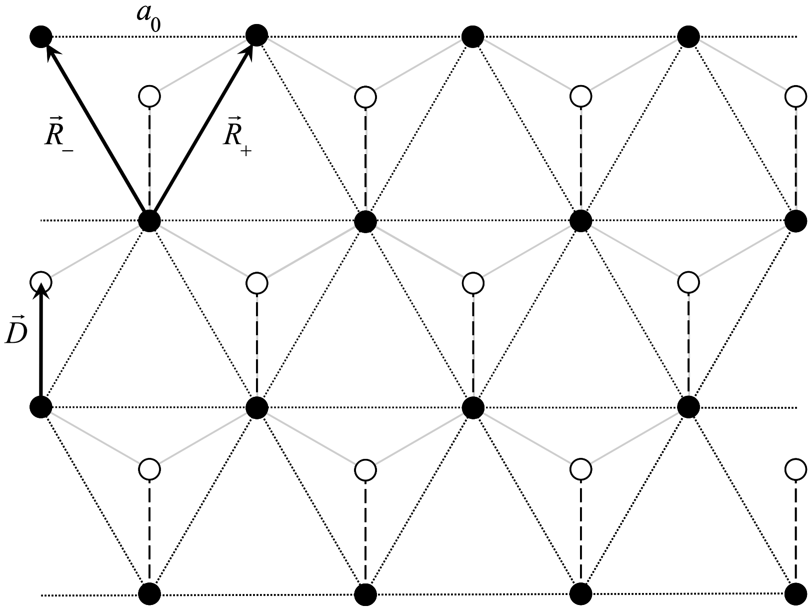


Figure 6: the 2-dimensional planar lattice of a layer of graphite. The honeycomb of carbon atoms can be described as a *Bravais lattice* of pairs of atoms arranged in an equilateral triangular array; the lattice constant a_0 is the length of a side of a triangle. The primitive vectors $\vec{R}_+$ and $\vec{R}_-$ form a basis which can locate the position of the lower atom in any pair (dark circles), and the vector $\vec{D}$ gives the relative position of the other atom in each pair (white circles).

The position of any lattice point can then be expressed as in equations (3), which are repeated again here:

$$
\begin{aligned}
&\vec{R}_{ij} = i\,\vec{R}_+ + j\,\vec{R}_- \quad ; \quad i, j \in \mathbb{Z} \\
&\vec{R}_+ = \tfrac{1}{2} a_0 \left(\sqrt{3}\,\hat{y} + \hat{x} \right); \vec{R}_- = \tfrac{1}{2} a_0 \left(\sqrt{3}\,\hat{y} - \hat{x} \right) \qquad (9) \\
&a_0 = 0.24612\,\text{nm for graphite}
\end{aligned}
$$

The dark atoms in figure 6 are at the lattice point positions; their partners are located at $\vec{D} = \left(a_0 / \sqrt{3} \right) \hat{y}$ with respect to each of the $\vec{R}_{ij}$.

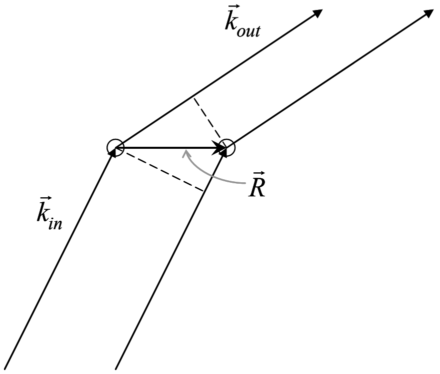


Figure 7: relation between wave vectors and the position vector of a lattice point. For constructive interference between the outgoing waves from the origin and from the lattice point at $\vec{R}$, it must be true that $\vec{R}\cdot(\vec{k}_{in}-\vec{k}_{out}) = 2\pi n$ for some integer n so that the waves are in phase.

Consider a plane wave impinging on a lattice of scattering sites. Choosing the phase of the wave at the origin as our phase reference, the phase at each lattice site would be (see figure 7):

$$\left(\phi_{ij}\right)_{in} = \vec{R}_{ij}\cdot\vec{k}_{in}$$

Each site radiates a wave in response to the incoming wave; since all lattice sites contain the same atom type, the phase difference between the response and the excitation is the same at all sites. Now consider a distant observer in the direction of an outgoing diffracted wave; for a diffracted wave to be radiated in this direction, the contribution of each lattice site to the total wave at the observer must have the same phase so that constructive interference occurs. Again referring to figure 7, the phase difference between the outgoing wave from the origin and the outgoing wave at each lattice site, if all lattice sites are oscillating in phase, would be:

$$\left(\phi_{ij}\right)_{out} = -\vec{R}_{ij}\cdot\vec{k}_{out}$$

The total phase difference between the origin and a lattice site would simply be the sum of these two contributions; for constructive interference to occur and a diffracted wave to be observed, the signals from all lattice sites must be in phase, so the phase difference must be a multiple of 2π:

$$\vec{R}_{ij}\cdot\left(\vec{k}_{in}-\vec{k}_{out}\right) = 2\pi\, n_{ij} \quad ; \quad n_{ij}\in\mathbb{Z} \tag{10}$$

The set of wave vectors $\vec{K}$ which satisfy

$$\vec{R}_{ij} \cdot \vec{K} = 2\pi\, n_{ij} \quad ; \quad n_{ij} \in \mathbb{Z} \tag{11}$$

form the *Reciprocal Lattice* of the crystal; from equation (10) we see that the difference in the incoming and outgoing wave vectors must be a member of the reciprocal lattice for a diffracted wave to be observed. This result was found by the German physicist Max von Laue, who won the Nobel Prize in 1914 for his discovery of X-ray diffraction in crystals; this requirement for diffraction is known as the *Laue condition*. William Henry Bragg and his son William Lawrence Bragg shared the Nobel in 1915 for their independent X-ray diffraction studies; their description of diffraction by the so-called *Bragg planes* of a crystal is equivalent to (10).

In the case of a single graphite layer, the reciprocal lattice vectors $\vec{K}$ lie in the *x*-*y* plane, so that (10) and (11) only require that the projection of $\vec{k}_{in} - \vec{k}_{out}$ into the *x*-*y* plane must equal a reciprocal lattice vector $\vec{K}$; in the case of a fully 3-dimensional crystalline structure such as aluminum the strict Laue condition must be satisfied.

The reciprocal of a Bravais lattice is also a Bravais lattice, which we state without proof — Reference 1 provides more detail on this subject. We shall show, however, that the Bravais lattice defined by equations (4) is indeed the reciprocal lattice of (9). We repeat (4) again here:

$$\begin{aligned}
&\vec{K}_{lm} = l\,\vec{K}_+ + m\,\vec{K}_- \quad ; \quad l,m \in \mathbb{Z} \\
&\vec{K}_+ = \tfrac{1}{2}K_0\left(\sqrt{3}\,\hat{x} + \hat{y}\right); \vec{K}_- = \tfrac{1}{2}K_0\left(\sqrt{3}\,\hat{x} - \hat{y}\right) \\
&K_0 = 4\pi / \left(\sqrt{3}\,a_0\right)
\end{aligned} \tag{12}$$

Since the vectors $\vec{K}_+$ and $\vec{K}_-$ are linearly independent, any reciprocal lattice vector $\vec{K}$ may be expressed as a linear combination of them. To show that the lattice is Bravais, we must show that *l* and *m* defined in (12) must be integers to satisfy equation (11), and that, in fact, any integer values for *l* and *m* satisfy (11):

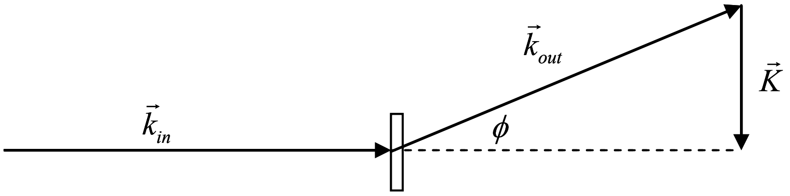
$$\begin{aligned}
\vec{R}_{ij} \cdot \vec{K}_{lm} &= \left(i\,\vec{R}_+ + j\,\vec{R}_-\right) \cdot \left(l\,\vec{K}_+ + m\,\vec{K}_-\right) \\
&= il\,\vec{R}_+ \cdot \vec{K}_+ + im\,\vec{R}_+ \cdot \vec{K}_- + jl\,\vec{R}_- \cdot \vec{K}_+ + jm\,\vec{R}_- \cdot \vec{K}_-
\end{aligned}$$

$$\vec{R}_+ \cdot \vec{K}_+ = -\vec{R}_- \cdot \vec{K}_- = 2\pi \ ; \ \vec{R}_+ \cdot \vec{K}_- = \vec{R}_- \cdot \vec{K}_+ = 0$$

$$\begin{aligned}
&\therefore\ \vec{R}_{ij} \cdot \vec{K}_{lm} = 2\pi\left(i\,l - j\,m\right) \ ; \ i,j \in \mathbb{Z} \\
&\therefore\ i\,l - j\,m = n \ ; \ n \in \mathbb{Z} \\
&\therefore\ l,m \in \mathbb{Z}
\end{aligned}$$

Where the last step follows by considering $i=1, j=0$ then $i=0, j=1$. So the vectors (12) do indeed define a Bravais lattice which is the reciprocal lattice of the crystal (9). Clearly, this lattice is geometrically similar to that in figure 6, but rotated 90°.

Consider the diffraction by the graphite target shown below, where the plane of the figure is the plane containing the incoming and outgoing electron waves, and the lattice plane is normal to this plane and the direction of the incoming wave.



$$K = k \sin\phi \tag{13}$$

The scattering is elastic, so the magnitudes of the incoming and outgoing wave vectors are the same. Compare this to equation (5) for diffraction by aluminum; in the limit of small scattering angles, equations (13) and (5) give the same result.

Lattice with a basis

The Bravais lattice sites defined in equations (9) include only half of the carbon atoms in the graphite layer. What about the other atoms located at position $\vec{D}$ with respect to each lattice site (refer again to figure 6)? When a lattice site contains a configuration of atoms rather than just a single atom, we define a Bravais lattice *with a basis*, the "basis" being a set of vectors and coefficients which define the relative positions and scattering efficiencies of the atoms comprising a single lattice site. Obviously, a crystal of a complicated molecule will have a complicated basis. In our case, though, the basis is quite simple: an identical scatterer at relative position $\vec{D}$.

The phase of the contribution of an atom located at $\vec{R}_{ij} + \vec{D}$ when $\vec{k}_{in} - \vec{k}_{out} = \vec{K}_{lm}$ is:

$$\begin{aligned}
&\left(\vec{R}_{ij} + \vec{D}\right)\cdot\vec{K}_{lm} = \vec{R}_{ij}\cdot\vec{K}_{lm} + \vec{D}\cdot\vec{K}_{lm} = 2\pi n + \vec{D}\cdot\vec{K}_{lm} \\
&\vec{D} = \left(a_0/\sqrt{3}\right)\hat{y} \\
&\vec{D}\cdot\vec{K}_{lm} = l\,\vec{D}\cdot\vec{K}_{+} + m\,\vec{D}\cdot\vec{K}_{-} = \tfrac{2}{3}\pi\left(l-m\right)
\end{aligned} \tag{14}$$

Since there are as many atoms at these sites as there are at the $\vec{R}_{ij}$, their contributions to the amplitude of the outgoing wave are equally important. Including their contributions, the relative intensity of an outgoing diffracted wave associated with the reciprocal lattice vector $\vec{K}_{lm}$ becomes:

$$\begin{aligned} I &= \tfrac{1}{4}\left|1+\exp\left[i\tfrac{2}{3}\pi\left(l-m\right)\right]\right|^2 = \cos^2\left[\tfrac{\pi}{3}\left(l-m\right)\right] \\ &= 1 \ : \ \left(l-m\right) \bmod 3 = 0 \\ &= \tfrac{1}{4} \ : \ \left(l-m\right) \bmod 3 \neq 0 \end{aligned} \tag{15}$$

So the diffracted waves will have two different intensities, depending on the l and m values of the associated reciprocal lattice vectors (this variation will be difficult to observe during this experiment, since the responses of the phosphorescent screen and your eyes are not very linear).