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EXPERIMENT 15

SOUND WAVES IN A CAVITY

A- INTRODUCTION.

In this experiment we will study the resonances in an acoustic cavity. The cavity you will be using is shown in Fig. 1.

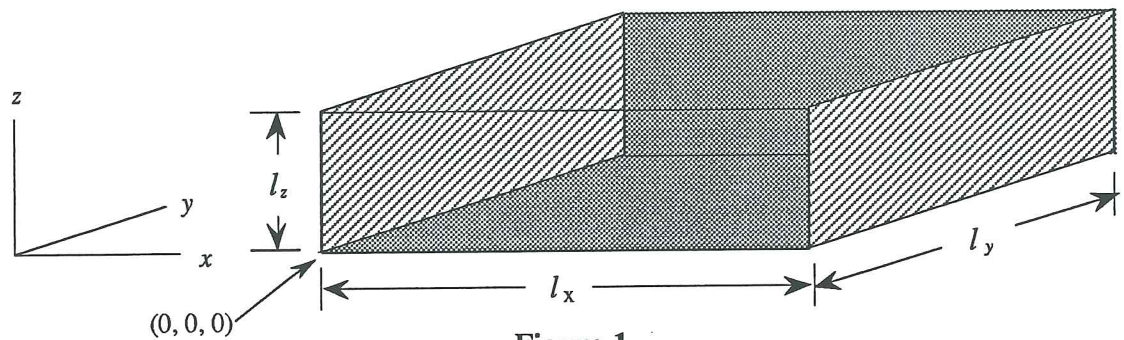


Figure 1
The cavity used in this experiment.

The three dimensional wave equations for sound waves are derived in the Appendix. If you wish to understand the origin of the various equations that you will be using in these notes read this appendix, but remember that its reading is optional. The wave equations for the pressure field p_e and displacement field $\vec{\chi}$ are the following:

$$\nabla^2 p_e = \frac{1}{v^2}\frac{\partial^2 p_e}{\partial t^2} \quad (1)$$

$$\nabla^2 \vec{\chi} = \frac{1}{v^2}\frac{\partial^2 \vec{\chi}}{\partial t^2} \quad (2)$$

Let's remember what these operators stand for:

$$\nabla^2 p_e = \frac{\partial^2 p_e}{\partial x^2} + \frac{\partial^2 p_e}{\partial y^2} + \frac{\partial^2 p_e}{\partial z^2}$$

and

$$\nabla^2 \vec{\chi} = \frac{\partial^2 \vec{\chi}}{\partial x^2} + \frac{\partial^2 \vec{\chi}}{\partial y^2} + \frac{\partial^2 \vec{\chi}}{\partial z^2}$$

Let's try the general plane wave solution

$$p_e = A e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{3}$$

Inserting Eq. 3 into Eq. 1 we obtain the condition for Eq. 3 to be a solution of the wave equation:

$$k_x^2 + k_y^2 + k_z^2 = \left(\vec{k}\right)^2 = k^2 = \frac{\omega^2}{\mathrm{v}^2} \tag{4}$$

Adding two waves traveling in the $+\vec{k}$ and $-\vec{k}$ with the same amplitude, taking the real part, and carefully choosing the real terms that satisfy the boundary conditions (see below), we obtain,

$$p_e = \mathrm{A} \times \cos(k_x x) \times \cos(k_y y) \times \cos(k_z z) \times \cos(\omega t) \tag{5}$$

Lets find the displacements $\vec{\chi}$ corresponding to the solution given by Eq. 5. Eq. A-VI in the Appendix is reproduced below

$$\vec{\nabla} p_e = -\rho_0 \frac{\partial^2 \vec{\chi}}{\partial t^2} \tag{A-23}$$

where ρ_0 is the average density of the medium. With this equation and the solution give by Eq. 5, we can find the corresponding displacements. We will, of course, assume that the displacements oscillate with the same frequency ω (See prelab 3). We obtain the following displacements:

$$\chi_x = -\frac{k_x}{\rho_0\, \omega^2} A \times \sin(k_x x) \times \cos(k_y y) \times \cos(k_z z) \times \cos(\omega t) \tag{6a}$$

$$\chi_y = -\frac{k_y}{\rho_0\, \omega^2} A \times \cos(k_x x) \times \sin(k_y y) \times \cos(k_z z) \times \cos(\omega t) \tag{6b}$$

$$\chi_z = -\frac{k_z}{\rho_0\, \omega^2} A \times \cos(k_x x) \times \cos(k_y y) \times \sin(k_z z) \times \cos(\omega t) \tag{6c}$$

The boundary conditions for the sound waves at the walls in this cavity are that the displacements, $\vec{\chi}$, <u>perpendicular</u> to each wall vanish (Prelab problem 1) :

$$\chi_x = 0 \quad \text{for } x = 0 \quad \text{and } x = l_x$$

$$\chi_y = 0 \quad \text{for } y = 0 \quad \text{and } y = l_y$$

$$\chi_z = 0 \quad \text{for } z = 0 \quad \text{and } z = l_z$$

Notice that the solutions given by Eqs. 6a-6c satisfy these conditions at x=0, y=0 and z=0. To satisfy the conditions at the other three walls we proceed as follows. Consider Eq. 6a, this displacement must vanish at x = l_x, For the sine to vanish at this wall $k_x l_x = n_x \pi$, where n_x must be an integer. From this condition we find the allowed values for k_x:

$$k_x = \frac{n_x \pi}{l_x} \qquad n_x = 0,1,2,..... \tag{7a}$$

Similarly we find the allowed values for k_y and k_z:

$$k_y = \frac{n_y \pi}{l_y} \qquad n_y = 0,1,2,..... \tag{7b}$$

$$k_z = \frac{n_z \pi}{l_z} \qquad n_z = 0,1,2,..... \tag{7c}$$

From the dispersion relation given by Eq. 4 and the conditions given by Eqs. 7a-7c we obtain the angular frequencies of the normal modes of this cavity.

$$\omega^2 = v^2 k^2 = v^2 \pi^2 \left[\left(\frac{n_x}{l_x}\right)^2 + \left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2 \right] \tag{8a}$$

Or in terms of the frequency f in Hz:

$$f^2 = \frac{v^2}{4} \left[\left(\frac{n_x}{l_x}\right)^2 + \left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2 \right] \tag{8b}$$

These are usually described as the (n_x, n_y, n_z) modes.

Also notice that from Eq. A-VI the corresponding conditions for p_e are that the pressure must be a maximum or a minimum at all the walls.

Figures 2 and 3 show the pressure distribution for $n_x = 1$ and for $n_x = 2$ as a function of x near $y = 0$ or $y = l_y$.

$$p_e = A\cos(k_x x) = A\cos\left(\frac{\pi}{l_x}x\right)$$

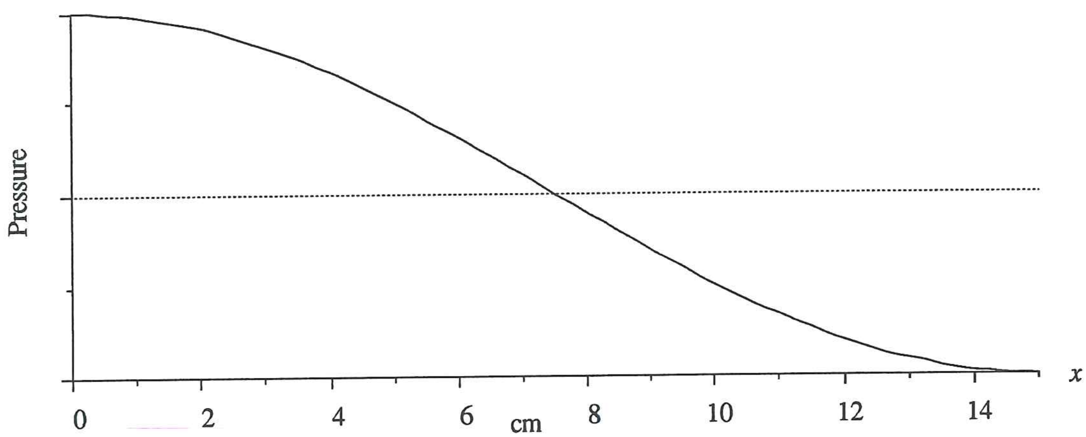


Figure 2
Pressure distribution along the x axis for $n_x = 1$

$$p_e = A\cos(k_x x) = A\cos\left(\frac{2\pi}{l_x}x\right)$$

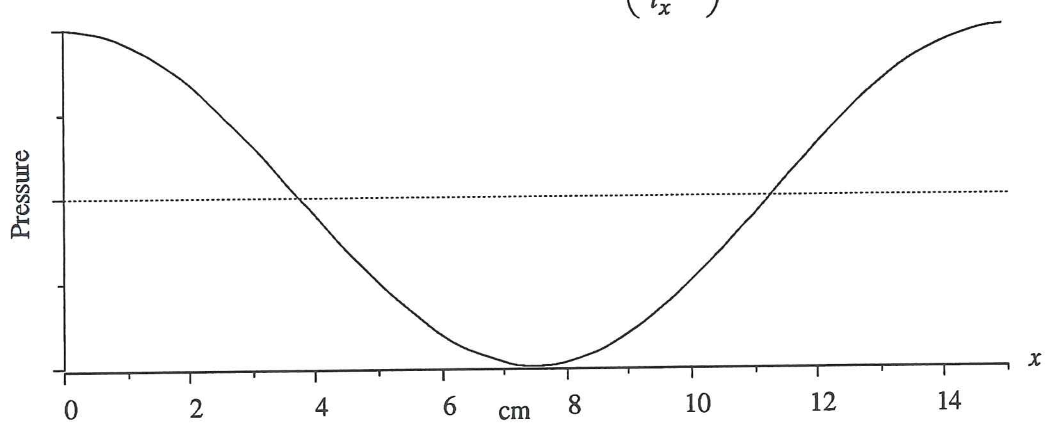


Figure 3
Pressure distribution along the x axis for $n_x = 2$

Figure 4 shows the nodal lines ($p_e = 0$) for various modes.

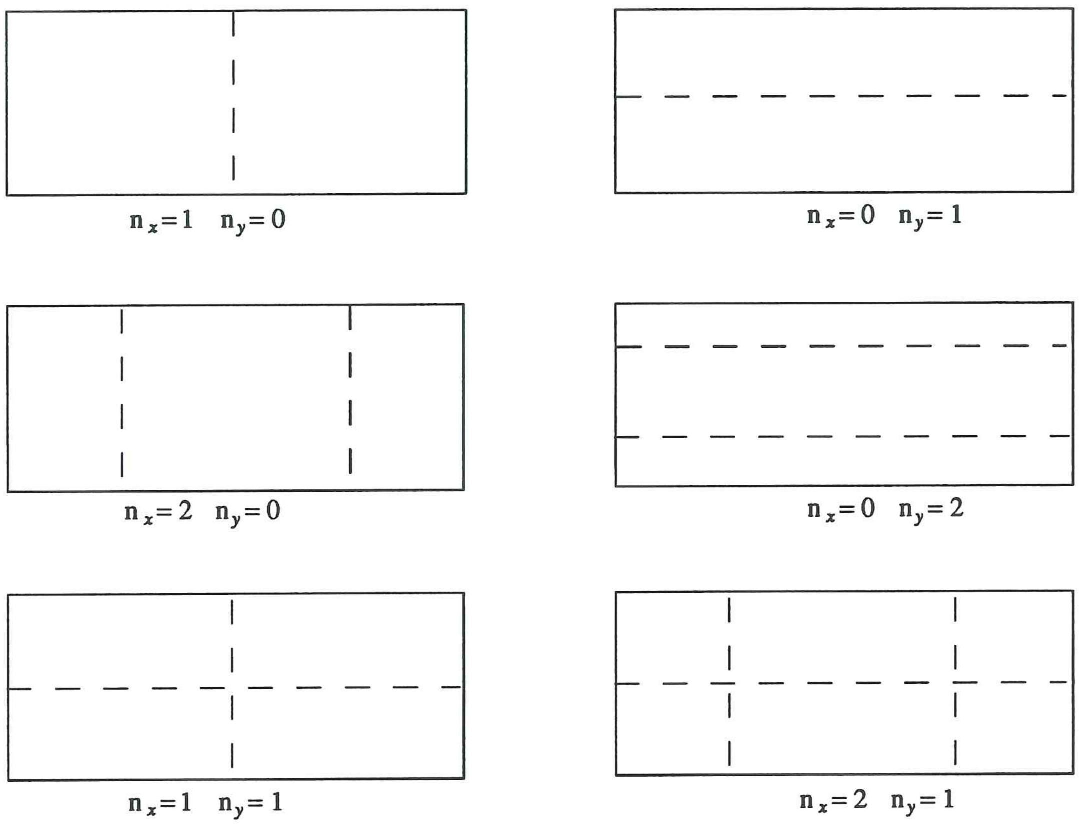


Figure 4
The nodal lines for a few modes.

Fig. 5a shows the pressure distribution in the x-y plane for $n_x = 1$ and $n_y = 1$ mode, and Fig. 5b shows this distribution for the $n_x = 2$ and $n_y = 1$ mode.

Figure 5a

Pressure distribution for:

$n_x = 1$ and $n_y = 1$

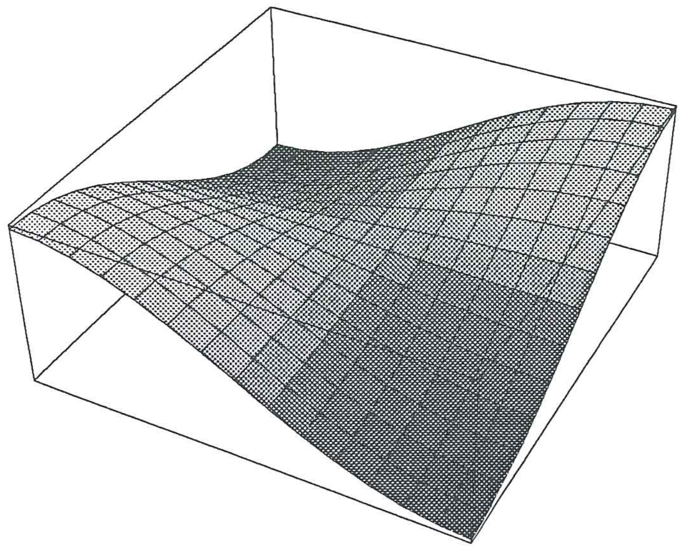
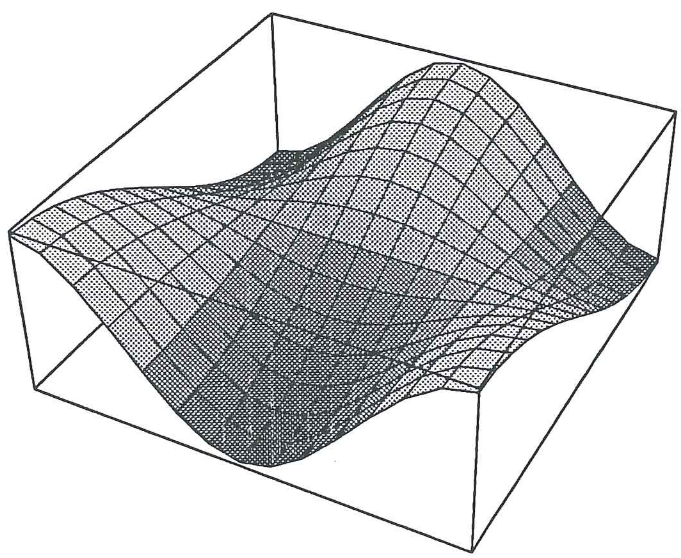


Figure 5b

Pressure distribution for:

$n_x = 2$ and $n_y = 1$



PRELAB PROBLEMS

1. Why is the boundary condition of the cavity such that the component of the air displacement $\vec{\chi}$ perpendicular to a wall must vanish at the wall?

2. Show that equation (5) is a solution to the wave equation (1) if the wavenumber components k_x, k_y, and k_z are related by equation (4).

3. Why does the boundary condition in problem 1 imply that the normal component of the pressure gradient ∇p_e also vanishes at a wall?

4. The cavity dimensions are approximately:

$$l_x = 15.24 \text{ cm},\ l_y = 11.51 \text{ cm},\ l_z = 3.18 \text{ cm}$$

   and the speed of sound at 24°C is 345.7 m/sec. Make a table listing the frequencies (in Hz) and mode indices (n_x, n_y, and n_z) for the 5 lowest frequency modes (resonances). Sketch in the $x-y$ plane the locations of the pressure nodal lines for these modes.

5. Calculate (in kilograms) the quantity m in equations (A15) and (A19). Assume air is 80% N_2 and 20% O_2.

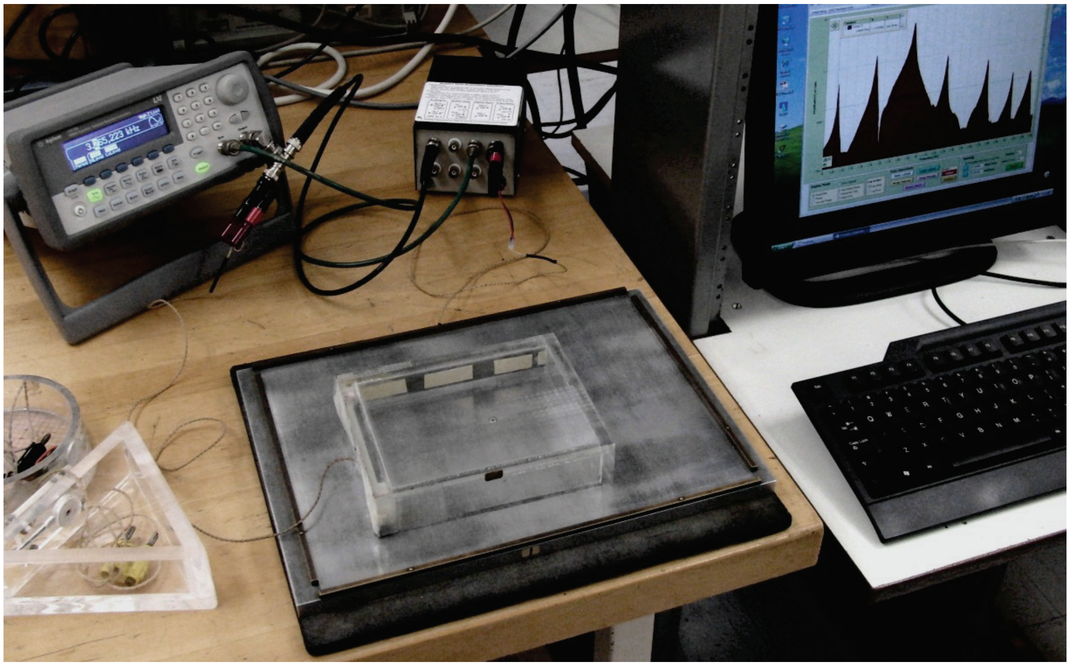


Figure 5: The experiment apparatus consists of a rectangular cavity which can be moved around on top of a metal base (which also serves as the bottom of the cavity). The driver transducer is mounted in the left vertical wall of the cavity near a corner and is connected to the signal generator's output. The receiver transducer is mounted in the center of the metal base plate. Its output is connected to either an external filter-amplifier or one that is built into the DAQ interface box (this latter arrangement is shown above). The filter-amplifier output is then measured by the computer DAQ. Cylindrical and triangular cavities are also available for additional studies.

THE EXPERIMENTAL SETUP

The apparatus is shown in figure 5. Identical, small microphones are used for both the driver and receiver transducers. The driver is near a corner of the cavity so that it is located far from any nodal lines for all but the very high-frequency resonant modes of the cavity. The receiver transducer is mounted in the center of the base plate. By moving the cavity around on the surface of the base plate, the receiver may be positioned at any point on the bottom wall of the cavity. Observing the receiver output as the cavity is moved will allow you to map out the nodal lines in the cavity for any particular resonant mode.

The *Frequency Response* application will provide a frequency spectrum of the cavity's resonant modes. After obtaining a detailed spectrum, you will configure the signal generator to output an appropriate *Tone Burst*; you may then use the *Transient Response* application to capture the time-domain (transient) response of the cavity to periodic excitations of a particular resonant mode.

PROCEDURE

Use calipers to measure the interior dimensions of the cavity. Take a few measurements along each edge so that you can determine the uncertainties in the measurements.

Record the lab's air temperature (why?).

Connect the output of the signal generator to the driver transducer and to the DAQ Ai-0 input.

Connect the receiver transducer to the Filter-Amplifier input. Connect the output of an external amplifier to the DAQ Ai-1 input. If an integrated filter-amplifier built into the DAQ interface box is used, then that amplifier's output is connected internally to the DAQ Ai-3 input; its external BNC output connector can then be used to connect an oscilloscope, if desired.

Connect the signal generator *Sync* output to the DAQ PFI-0 input, as usual.

Position the cavity on the base plate so that the receiver transducer is in a corner of the cavity. This position should be an anti-node for all of the cavity resonances (why?).

Launch the *Frequency Response* application and then set the signal generator output amplitude to about 1 Volt (peak-peak). Configure the program to use the appropriate DAQ connection for the response waveform (Ai-1 if an external amplifier is used; Ai-3 if using a built-in DAQ filter-amplifier).

You should find a strong resonance at about 1.9 kHz. What mode is this? Adjust the DAQ gains as necessary and sweep the frequency from about 20% below the lowest expected mode frequency to about 5 kHz. You should find several resonances. Make sure you get good resolution about each resonant peak so that you can accurately determine its frequency. The spectrum on the computer display in Figure 5, for example, shows the first several resonances of the rectangular cavity. Note that the other resonant peaks are generally much weaker than the 1.9 kHz one.

Estimate the Q of each of the first 3 modes by examining the frequency widths of the resonant peaks. What do you think may be some of the dominant loss mechanisms which cause the energy in the cavity to dissipate and limit the Q?

Tune the signal generator to each of the first 5 resonant frequencies and move the cavity around so that you can map the nodal line positions for each of these modes. Do they match what you predicted in your prelab problem solutions?

Are the various high-frequency resonant peaks all well-resolved or are there pairs of peaks which are very close together? Such pairs of resonances are called *accidental degeneracies* in the system's response. Based on the frequencies of a nearly degenerate pair of resonances, determine the expected nodal pattern for each resonance of the pair. What is the actual nodal structure you observe? How does the shape of the frequency response of these resonances change if you move the receiver to a different corner of the cavity and take another frequency response sweep of the peaks?

Transient Response Measurements

To measure the transient response of a single mode of the cavity, you must inject energy mostly into that mode. To accomplish this you must use a *tone burst*: the signal generator's output produces a sinusoid at the mode resonant frequency for a few Q cycles, and then the output is abruptly turned off for another few Q periods of the resonant frequency. The cavity then "rings down" at mostly that frequency, since most (but not all!) of the energy injected by the generator was stored in that mode.

Use the *Transient Response* application to capture the transient response of the cavity at the first and second resonances. First tune the signal generator *Sine* output to a mode's resonant frequency and set its output amplitude to a few volts. Then set the signal generator to *Tone Burst* and set up the *tone burst* number of cycles and burst period.

The signal generator *Sync* output rises when the tone burst starts and falls when the burst ends. You should therefore configure the *Transient Response* program to trigger on the *falling* edge of the trigger signal.

Does the amplitude of the decaying sine wave decrease monotonically, or do you see some sort of "beat" in the amplitude as it decays? Estimate the frequency of the beat, if any. What could be causing this? If you change the position of the receiver transducer, does the beat amplitude change? Can you find a position where the beat disappears? What is going on?

Estimate the Q from the time constant of the overall decay.

Investigate the resonant modes of at least one of the alternate (cylindrical or triangular) cavities. There are Mathematica notebooks which investigate the mathematics of the modes of these cavities on the Physics 6 website:

http://www.sophphx.caltech.edu/Physics_6/Mathematica Notebooks/Sound waves in a cavity/

DATA ANALYSIS

The resonances you observe are *cavity resonances*. The theory of the detailed shape of the frequency spectrum for this two-dimensional cavity is complicated, although the theory predicting the resonant mode wavenumbers is not. A theory of the response of a finite-Q, one-dimensional cavity is significantly simpler and is presented in *General Appendix A* of the lab notes.

As in *General Appendix A*, the shape of an intensity peak close to a resonance is approximately *Lorentzian* (+ a linear background). The intensity is the square of the wave amplitude, so you should square the frequency response gain magnitude before attempting a Lorentzian fit to determine the Q (ω_0/γ) of the resonance. Compare the Q obtained to that from your transient response data for the appropriate resonances.

Create a data file of the several mode resonant frequencies (as Y) versus their wavenumbers (as X) derived from the mode indices and the cavity dimensions (equation (8b)). What should be the functional form of the fit as predicted by the theory, equation (8b)?

Is the air nondispersive over this frequency range? What is the speed of sound (with uncertainty)? How could the uncertainties in your measurements of the cavity dimensions affect the uncertainty in the speed of sound? Are these uncertainties systematic, or should they be included as error bars on the individual mode wavenumber X values? Why or why not? If they are systematic, how will you determine their effect of your uncertainty in the speed of sound?

APPENDIX

THE WAVE EQUATION FOR SOUND

The wave equation in one dimension

We will be following Feynman Vol. I sections 47:2-5. In the following derivation of the wave equation for sound it is very important to realize that the changes of pressure and density associated with a sound wave are exceedingly small compared with the pressure and density of air at STP (Standard Temperature and Pressure). Some important numbers follow.

Numbers.

The loudness of a sound (its intensity) is measured in decibels (db) and is defined by

$$I(db) = 20 \times \log_{10}\left(\frac{p_e}{p_{ref}}\right) \tag{A1}$$

Where p_e is the pressure change due to the sound and p_{ref} is the refrence pressure (chosen at some international meeting many years ago),

$$p_{ref} = 2\times10^{-4}\ \text{dynes}/\text{cm}^2 \approx 2\times10^{-10}\ \text{atmospheres (at STP)}^*$$

The Concord at take off produces a sound at the ground of around 110 db, some Rock and Roll concerts go as high as 120 db. 120 db is defined as the pain threshold, however, Rock and Roll seems to be painful even at 0 db. Let's calculate p_e for a 120 db sound.

$$\log_{10}\left(\frac{p_e}{p_{ref}}\right) = \frac{120}{20} = 6$$

or

$$\frac{p_e}{p_{ref}} = 10^6 \qquad or \qquad p_e = 10^6 P_{ref}$$

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* With this choice of p_{ref} the p is not the peak pressure of the sound wave but the "root-mean-square" pressure, which is $1/\sqrt{2}$ times the peak pressure.

or

$$p_e \approx 2\times10^{-4} \text{ atmospheres}$$

Therefore the pressure changes due to sound (and, therefore, the density changes) are very, very small compared to the ambient pressure (or density). Other amazingly small numbers: in normal conversation, at one meter away, the sound intensity is about 60 db, the amplitude of the oscillations (which will be denoted by χ) of the air corresponding to this intensity is only 10^{-6} cm! That is, the ear drum oscillates with an amplitude of less than 100 atomic diameters! For the pressure p_{ref}, which is supposed to be the threshold of hearing, the ear drums oscillates with an amplitude of about 1/10 of an atomic diameter!

Let's define some symbols. Let's expand the pressure in a power series around the ambient pressure p_0 keeping only the first order term

$$p = p_0 + \Delta p$$

where Δp is the change in the ambient pressure due to the sound wave. Defining

$$p_e = \Delta p$$

then

$$p = p_0 + p_e \tag{A2}$$

Now we do the same with the density. Let's expand the gas density in a power series around the ambient density ρ_0 keeping only the first order term

$$\rho = \rho_0 + \Delta\rho$$

where $\Delta\rho$ is the change in the ambient density due to the sound wave. Defining

$$\rho_e = \Delta\rho$$

then

$$\rho = \rho_0 + \rho_e$$

The changes of ρ_e / ρ_0 are as small as the pressure changes, that is, of the order of 10^{-4}.

Now we start the real work.

Consider p as a function of ρ, and expand p in a power series around p_0 keeping only the first order term:

$$p = p_0 + \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0} \times \Delta\rho = p_0 + \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0} \times \rho_e \tag{A3}$$

From Eqs. A2 and A3 we obtain

$$p_e = \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0} \times \rho_e$$

or

$$\boxed{\begin{array}{c} \text{defining} \quad \mathrm{v}^2 = \left.\dfrac{\partial p}{\partial \rho}\right|_{\rho_0} \\ \\ p_e = \mathrm{v}^2 \rho_e \end{array}} \tag{A4}$$

As we will find out, v will turn out to be the speed of sound.

<u>Plane Waves</u>. In this lecture we will only deal with <u>plane waves</u>, these are waves that propagate in <u>one</u> direction only, in the $\pm\, x$ directions in our case, and are therefore functions of x and t only. Consider a sound wave propagating in the $+x$ direction and imagine two very thin plastic sheets (with negligible mass) located at x and at $x + \Delta x$ (think of x and $x + \Delta x$ as labeling these two sheets). Due to the sound wave these two sheets move back and fore. Let's denote their displacements at time t by $\chi(x,t)$ for sheet x, and

by $\chi(x+\Delta x,t)$ for sheet $x+\Delta x$, as shown in Fig. 1. Notice: x or $x+\Delta x$ are not functions of t.

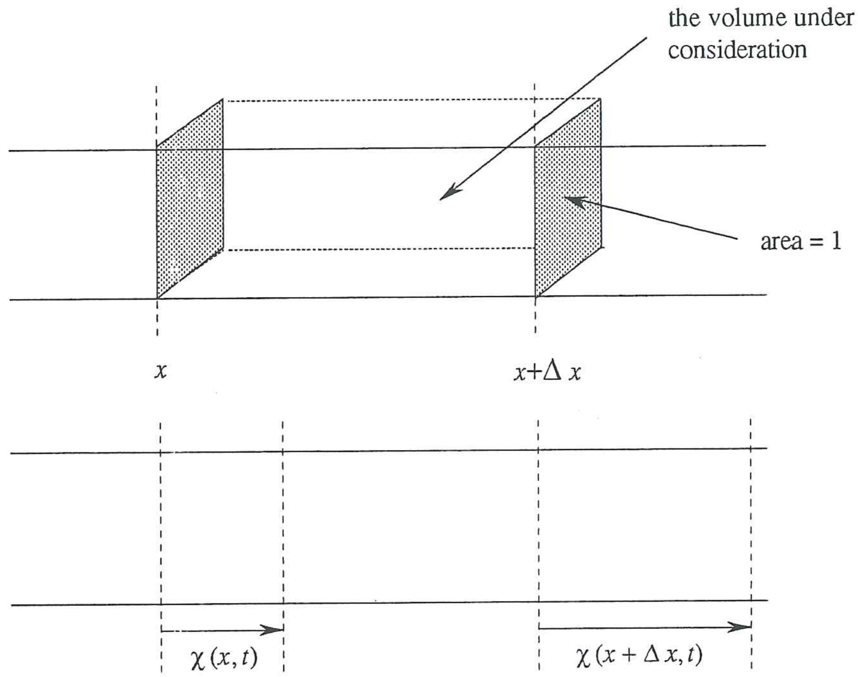

Figure 1
A plane sound wave propagating on the x direction.

Introduce the Physics. Consider the volume of gas enclosed by the two plastic sheets with unit cross section.

1– Conservation of Mass. The mass of gas in this volume is constant:

$$\rho_0 \Delta x = (\rho_0+\rho_e)[\Delta x+\chi(x+\Delta x,t)-\chi(x,t)]$$

Now divide by Δx and let $\Delta x \to 0$

$$\rho_0 = (\rho_0+\rho_e)\left(1+\frac{\partial \chi}{\partial x}\right)$$

or

$$\rho_e+\rho_0\frac{\partial \chi}{\partial x}+\rho_e\frac{\partial \chi}{\partial x}=0 \qquad \text{(A5)}$$

As it was mentioned at the beginning of these notes, the displacement of the gas (our plastic sheets) due to a sound wave is exceedingly small and for this reason the term $\rho_e \frac{\partial \chi}{\partial x}$ is much smaller than the other two terms in Eq. A5. Neglecting this term we obtain from Eq. A5

$$\boxed{\rho_e = -\rho_0 \frac{\partial \chi}{\partial x}} \qquad \text{from conservation of mass} \quad \text{(A6)}$$

2– $\underline{F} = \underline{ma}$. Let's now look at the external forces acting in the x direction on our volume, as shown in Fig. 2.

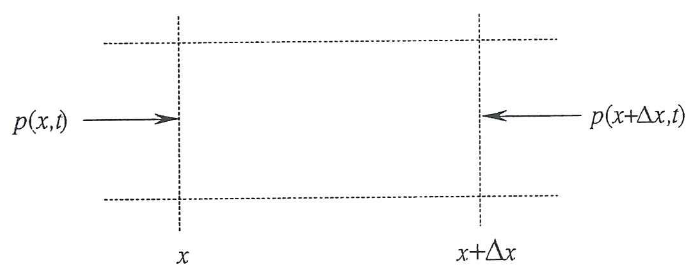

Figure 2
The external forces acting on the volume under consideration.

The net force in the x direction is given by

$$f_x = p(x,t) - p(x+\Delta x,t)$$

Using Eq.A1 we obtain

$$f_x = p_e(x,t) - p_e(x+\Delta x,t) \qquad \text{(A7)}$$

This force produces an acceleration on the mass between x and $x + \Delta x$ given by

$$f_x = \underbrace{\rho_0 \Delta x}_{\text{mass}} \underbrace{\frac{\partial^2 \chi}{\partial t^2}}_{\text{acceleration}} \qquad \text{(A8)}$$

From Eqs. A7 and A 8 we obtain

$$p_e(x,t) - p_e(x+\Delta x,t) = \rho_0\,\Delta x \frac{\partial^2 \chi}{\partial t^2}$$

dividing this equation by Δx and rearranging

$$\frac{p_e(x+\Delta x) - p_e(x)}{\Delta x} = -\rho_0 \frac{\partial^2 \chi}{\partial t^2}$$

taking the limit as $\Delta x \to 0$

$$\boxed{\frac{\partial p_e}{\partial x} = -\rho_0 \frac{\partial^2 \chi}{\partial t^2}} \quad \underline{\text{form } F = m\,a} \qquad \text{(A9)}$$

Let's collect these three important equations

$$\boxed{\begin{array}{ll} p_e = \mathrm{v}^2 \rho_e & \text{I} \\ \text{with } \mathrm{v}^2 = \left.\dfrac{\partial p}{\partial \rho}\right|_{\rho_0} & \\ \rho_e = -\rho_0 \dfrac{\partial \chi}{\partial x} & \text{II} \\ \dfrac{\partial p_e}{\partial x} = -\rho_0 \dfrac{\partial^2 \chi}{\partial t^2} & \text{III} \end{array}}$$

<u>The wave equation in one dimension.</u>

To obtain the wave equation for the the sound pressure we proceed as follows. Take the partial derivative of Eq. III in relation to x

$$\frac{\partial^2 p_e}{\partial x^2} = -\rho_0 \frac{\partial^2}{\partial t^2}\left(\frac{\partial \chi}{\partial x}\right)$$

insert Eq. II

$$\frac{\partial^2 p_e}{\partial x^2} = \frac{\partial^2 \rho_e}{\partial t^2}$$

and with Eq. I we obtain

$$\boxed{\frac{\partial^2 p_e}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 p_e}{\partial t^2}} \qquad \text{(A10)}$$

To obtain the wave equation for the displacement associated with a sound wave we proceed as follows. From Eqs. I and II we obtain

$$\rho_0 \frac{\partial^2 \chi}{\partial x^2} = -\frac{1}{\mathrm{v}^2}\frac{\partial p_e}{\partial x} \qquad \text{(A11)}$$

inserting Eq. III into Eq. A10

$$\boxed{\frac{\partial^2 \chi}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 \chi}{\partial t^2}} \qquad \text{(A12)}$$

Eqs. A10 and A12 are the wave equations that we have seen earlier in this course. We know that its most general solutions are waves that travel in the $\pm x$ direction with velocity v. Therefore, v is indeed the speed of sound.

The velocity of sound. The velocity of sound is given by

$$v^2 = \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0} \tag{A13}$$

Air at STP behaves as an ideal gas, therefore

$$p\,V = N\,k\,T \tag{A14}$$

where p is the pressure, T the temperature, k the Boltzman's constant, and N the number of molecules in the volume V. The gas density is given

$$\rho = \frac{mN}{V} \tag{A15}$$

where m is the molecular mass (or in a mixture of gases, the average molecular mass). Using Eq. A14, Eq. A15 can be rewritten as

$$p = \frac{1}{m}\rho\,kT \tag{A16}$$

Now we have to decide the type of process that is involved in the compressions and expansions associated with a sound wave.

Newton's (very trivial) Mistake. Newton made the very logical assumption that the process is isothermal, that is, that the air temperature does not change. Differentiating Eq. 16 holding T constant we obtain

$$v^2 = \frac{kT}{m} \qquad \text{(a bit wrong)} \tag{A17}$$

The correct process. It turns out that, for a large range of frequencies (including the whole range of audio frequencies), the compressions and expansions are so fast that there is no time for the air to exchange heat with the surrounding air, therefore we are dealing with an adiabatic process. For an adiabatic process the pressure and the volume of an ideal gas change according to Eq. A18:

$$pV^{\gamma} = \text{constant} \qquad \text{(A18)}$$

where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. Dividing Eq. A18 by $(mN)^{\gamma}$, a constant, we obtain

$$p\left(\frac{V}{mN}\right)^{\gamma} = p\rho^{-\gamma} = A = \text{constant}$$

or

$p = A\rho^{\gamma}$

Using this expression in Eq. 4

$$\text{v}^2 = \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0} = \gamma A \rho_0^{\gamma-1} = \gamma \frac{A\rho_0^{\gamma}}{\rho_0} = \gamma \frac{p_0}{\rho_0}$$

And using Eq. A16 we get

$$\text{v}^2 = \gamma \frac{kT}{m} \qquad \text{(A19)}$$

For air at 0-100 °C $\gamma = 1.4$. Comparing Eq. A17 with Eq. A19 we see that Newton's mistake was only $\sqrt{\gamma}$, which is only 18 % from the correct value.

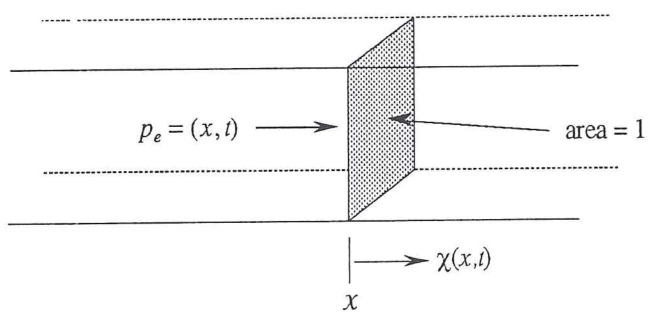

Figure 3
The massless plastic sheet oscillating back and fore due to a sound wave.

Sound Intensity. The *intensity* of a sound wave is defined as the time average of the energy crossing a unit area normal to the direction of propagation of the wave. Let's consider a sound wave propagating in the +*x* direction. Consider one of our massless plastic sheets, as shown in Fig. 3, and calculate the work done by the gas on the left of this sheet on the gas on the right. In a time *dt* this work is the force per unit area (the pressure) times the displacement $d\chi$,

$$dW = p_e(x,t)\,d\chi$$

or

$$\frac{dW}{dt} = p_e(x,t)\frac{d\chi}{dt}(x,t)$$

and taking the time average we obtain the *intensity*

$$I = \langle p_e(x,t)\frac{d\chi}{dt}(x,t)\rangle$$

The wave equation in three dimensions

Let's generalize the various equations to three dimensions. Fig. 4 shows the volume of gas that will be under our consideration.

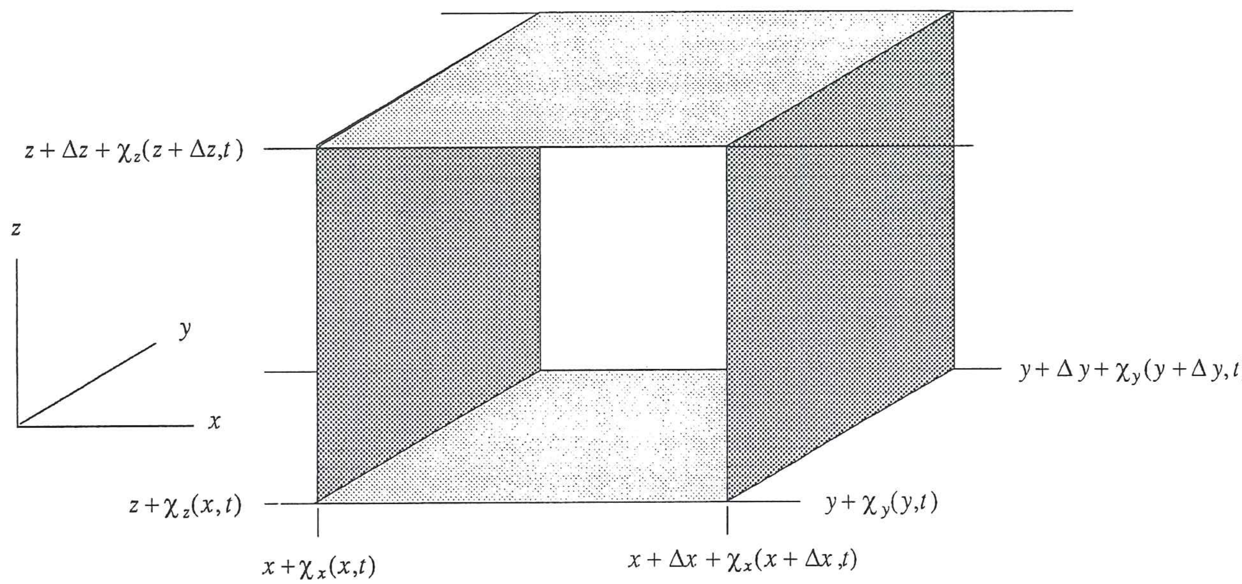

Figure 4
The volume under consideration.

1- Conservation of Mass .

$$\begin{aligned}\rho_0\Delta x\Delta y\Delta z = (\rho_0+\rho_e)&\times[\Delta x+\chi_x(x+\Delta x)-\chi_x(x)]\\ &\times[\Delta y+\chi_y(y+\Delta y)-\chi_y(y)]\\ &\times[\Delta z+\chi_z(z+\Delta z)-\chi_z(z)]\end{aligned}$$

Neglecting second and third order terms we get:

$$\begin{aligned}\rho_e\Delta x\Delta y\Delta z = -\rho_0\times\{&\Delta x\Delta y[\chi_z(z+\Delta z)-\chi_y(z)]\\ &+\Delta y\Delta z[\chi_x(x+\Delta x)-\chi_x(x)]\\ &+\Delta z\,\Delta x[\chi_y(y+\Delta y)-\chi_y(y)]\}\end{aligned}$$

Dividing by $\Delta x\ \Delta y\ \Delta z$ and letting $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta z \to 0$ we get

$$\rho_e = -\rho_0\left(\frac{\partial \chi_x}{\partial x} + \frac{\partial \chi_y}{\partial y} + \frac{\partial \chi_z}{\partial z}\right)$$

or

$$\boxed{\rho_e = -\rho_0 \vec{\nabla} \cdot \vec{\chi}} \quad \text{from conservation of mass} \quad (A20)$$

Eq. A20 is the three dimensional equivalent of Eq. A6.

2- F = ma.

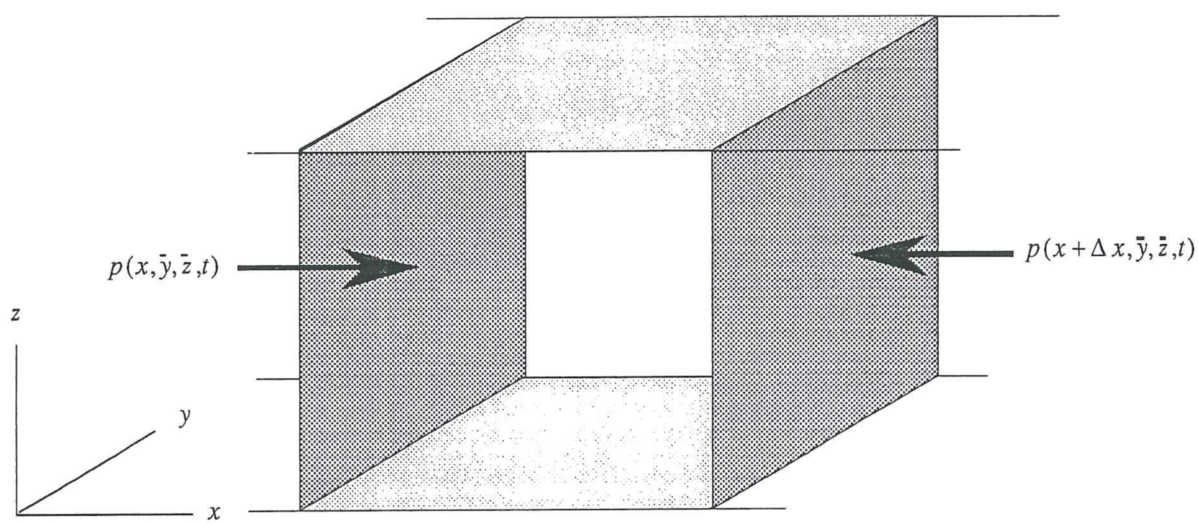

Figure 5
The force in the x direction on the volume under consideration.

First let's calculate the net force in the x direction on our volume as shown in Fig. 5.

$$f_x = \Delta y \Delta z\left[p(x,\bar{y},\bar{z},t) - p(x+\Delta x,\bar{y},\bar{z},t)\right]$$

and using Eq. A2

$$f_x = \Delta y \Delta z \left[p_e(x,\bar{y},\bar{z},t) - p_e(x+\Delta x,\bar{y},\bar{z},t) \right]$$

Dividing by Δx Δy Δz and letting $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta z \to 0$ we get

$$\frac{f_x}{\Delta x \Delta y \Delta z} = -\frac{\partial p_e}{\partial x} \tag{A21}$$

And now let's calculate $m\,a_x$.

$$m a_x = \rho_0 \Delta x \Delta y \Delta z \frac{\partial^2 \chi_x}{\partial t^2}$$

or

$$\frac{m a_x}{\Delta x \Delta y \Delta z} = \rho_0 \frac{\partial^2 \chi_x}{\partial t^2} \tag{A22}$$

And since $\qquad f_x = m a_x$

Eqs. A21 and A22 yield

$$\frac{\partial p_e}{\partial x} = -\rho_0 \frac{\partial^2 \chi_x}{\partial t^2}$$

And generalizing this equation to three dimensions

$$\boxed{\vec{\nabla} p_e = -\rho_0 \frac{\partial^2 \vec{\chi}}{\partial t^2}} \quad \underline{\text{from}} \quad \underline{F} = \underline{m}\ \underline{a} \tag{A23}$$

Eq. A21 is the three dimensional equivalent of Eq. A9.

Let's collect these important equations.

$$p_e = \mathrm{v}^2 \rho_e \qquad IV$$

$$\textit{with } \mathrm{v}^2 = \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}$$

$$\rho_e = -\rho_0 \vec{\nabla} \cdot \vec{\chi} \qquad V$$

$$\vec{\nabla} p_e = -\rho_0 \frac{\partial^2 \vec{\chi}}{\partial t^2} \qquad VI$$

Notice that p_e and ρ_e are still scalar fields while $\vec{\chi}$ is now a vector field.

<u>The wave equation in three dimensions</u>.

Let's take the divergence of Eq. VI

$$\nabla^2 p_e = -\rho_0 \frac{\partial^2 \vec{\nabla} \cdot \vec{\chi}}{\partial t^2}$$

inserting Eq. V

$$\nabla^2 p_e = \frac{\partial^2 \rho_e}{\partial t^2}$$

and finally inserting Eq. VI we obtain

$$\boxed{\nabla^2 p_e = \frac{1}{\mathrm{v}^2} \frac{\partial^2 p_e}{\partial t^2}} \qquad \text{(A24)}$$

Eq. A24 is the equivalent to Eq. A10.

Now let's take the gradient of Eqs. IV and V.

$$\vec{\nabla} p_e = \mathrm{v}^2 \vec{\nabla} \rho_e$$

$$\vec{\nabla} \rho_e = -\rho_0 \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\chi}\right)$$

Using these last equations

$$\vec{\nabla} p_e = -\mathrm{v}^2 \rho_0 \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\chi}\right)$$

Inserting Eq. VI

$$\boxed{\vec{\nabla}\left(\vec{\nabla} \cdot \vec{\chi}\right) = \frac{1}{\mathrm{v}^2} \frac{\partial^2 \vec{\chi}}{\partial t^2}} \tag{A25}$$

and this is the equation equivalent to Eq. A12.

Let's find a general plane wave solution to Eq. A25

$$\vec{\chi} = \vec{A} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \tag{A26}$$

Inserting Eq. A26 into Eq. A25 we obtain:

$$(\vec{k} \cdot \vec{\chi})\vec{k} = \frac{\omega^2}{\mathrm{v}^2} \vec{\chi} \tag{A27}$$

Which means that χ points in the direction of k (that makes good sense! Sound waves are longitudinal waves). Therefore we can rewrite our very general solution as

$$\vec{\chi} = \hat{B}\,\vec{k}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{A28}$$

and Eq. A27 can be rewritten as

$$(\vec{k}\cdot\vec{\chi})\vec{k} = \frac{\omega^2}{\mathrm{v}^2}\vec{\chi} \tag{A29}$$

Which requires that

$$k^2 = \frac{\omega^2}{\mathrm{v}^2}$$

Which is the dispersion relation for sound waves.