Experiment 2

A SIMPLE RESONANT SYSTEM

INTRODUCTION

Resonant behavior can be very interesting. Providing a sinusoidal stimulus to a physical system at a frequency near a resonant frequency may result in a *dramatic* change in the system's response — a change which may result in qualitatively different behavior unpredictable from the system's response at frequencies far from resonance. Conversely, an abrupt, transient stimulus to the system may cause the system to exhibit a prolonged response consisting of a superposition of various oscillations at frequencies near the system's resonant frequencies. By examining the response of the system to such steady-state and transient stimuli, the physicist gathers evidence regarding the internal degrees of freedom of the system (the system's *normal modes*) and may use this knowledge to construct or test theoretical models of the system's underlying structure.

The measured response of a system as a function of the frequency of a sinusoidal stimulus provides a ***spectrum*** of the system's response. Much of physics research is occupied with the gathering of ever more accurate and/or higher-frequency spectra of various fundamental physical systems. It is hoped that such experiments may discover behavior which is not adequately described by current theory and may provide valuable clues as to how theory may be extended or refined.

In this experiment you will investigate the behavior of a seemingly simple system which may be modeled fairly accurately by the theory of a *damped harmonic oscillator* — the simplest example of a system with a resonant response. The system is an electrical circuit consisting of a series combination of a resistor, inductor, and capacitor (a "*series RLC*" circuit). You will accurately measure the circuit's response to both steady, sinusoidal and short, impulsive voltage stimuli by using a fairly sophisticated, computer-controlled data acquisition system. You will analyze the data in part by using the *CurveFit* data analysis package with *Mathematica*®. The purpose of the experiment is for you to provide a detailed, critical evaluation of the simple theory presented to explain the circuit's behavior and to suggest extensions to the theory, if necessary, to explain discrepancies between your results and the theoretical model's predictions.

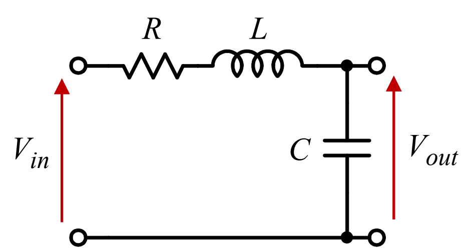


Figure 1: The basic resonant system. The value of the inductance L is either approximately 10 *millihenry* (mH, 10^{-3} henry) or 100 mH; the capacitance C is approximately 10 *nanofarad* (nF, 10^{-9} farad). The resistance R may be varied in value as a part of the experiment, or it may have a fixed value. The input voltage V_{in} is controlled by the experimenter; V_{out} is the measured response of the system.

Figure 1 shows a schematic diagram of the series *RLC* circuit you will examine. A signal generator, oscilloscope, and computer data acquisition (DAQ) system, along with various cables and adapters, then complete the experimental setup (Figure 2).

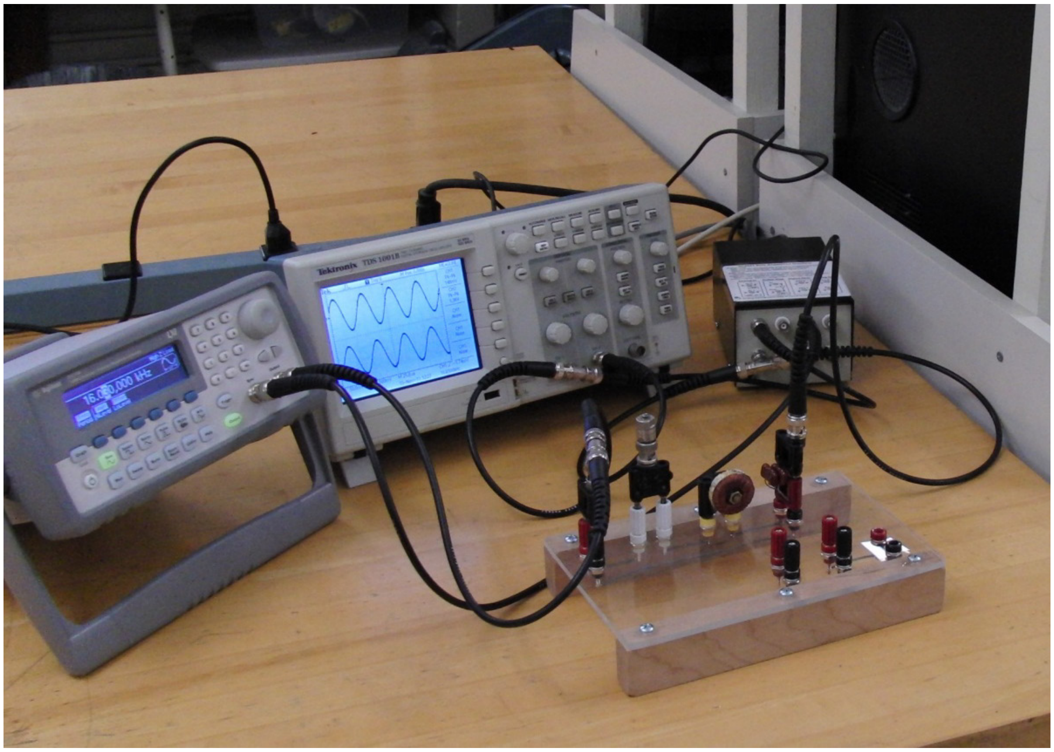


Figure 2: A typical setup which also includes an oscilloscope to monitor the time-varying input and output voltage signals. The *RLC* circuit is in the front right of the photo; several coaxial cables and adapters connect the system to the experimenter's instrumentation. A sinusoidal stimulus is being applied using the signal generator at left; the stimulus and response voltages are input to a computer-controlled data acquisition system (DAQ) using the interface box to the right of the oscilloscope. The computer may control the frequency of the signal generator using its USB interface, allowing the computer to perform frequency sweeps of V_{in} and measure a frequency-domain spectrum of the system response.

The "resistance" R in the circuit of Figure 1 is really a combination of two independent parts. One part of R is the resistance of a physical component that you will insert into the circuit, which will have a value of a few 10's of ohms. The other part of R represents the energy dissipation (loss) in the inductor's wire coil and in its ferromagnetic core. Obviously this second piece of R may not be directly accessible to you, but is rather a part of the theoretical description of the behavior of the circuit. Your data will allow you to assign a value for this extra bit, which will be referred to as R_L, since it is dominated by electrical energy losses in the inductor (L is the traditional electrical engineering designator of an inductor).

THEORY

The simple theory we will examine assumes that the R, L, and C values for the circuit of Figure 1 are all ***real, positive constants*** which are independent of time, voltage (or current), and frequency. The theory also assumes that Kirchhoff's circuit laws apply to our experiment (this implies that the physical size of our system is so small that at the frequencies we are using the wavelengths of any electromagnetic waves we generate are very much larger than our circuit). Finally, we assume that the currents flowing into our measuring devices may be neglected, so that the same current $I(t)$ flows through the series-connected R, L, and C (see Figure 3). Kirchhoff's voltage law then requires that the sum of the voltages across the components equals the input voltage, $V_{in}(t)$ as in Figure 3.

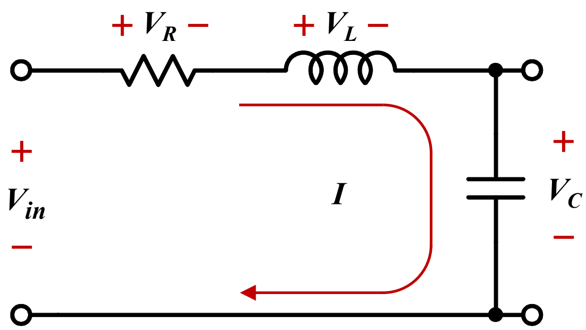


Figure 3: The series *RLC* circuit identifying the circuit's current *I* and various voltages, all of which are functions of time, *t*. The polarity of a positive voltage value is indicated for each voltage variable. The same current flows from the source through all circuit elements, since it is assumed that I_{out} vanishes. A positive current value indicates a current flow in the direction of the arrow. Kirchhoff's voltage law requires that $V_{in}(t) = V_R(t) + V_L(t) + V_C(t)$ and that $V_{out}(t) = V_C(t)$.

Review this experiment's *Appendix A* starting on page 21. The relationships among the various voltages and the current in Figure 3 are:

$$\begin{gathered} V_{in}(t) = V_R(t) + V_L(t) + V_C(t) \\ V_R(t) = R\,I(t); \quad V_L(t) = L\tfrac{d}{dt}I(t); \quad C\tfrac{d}{dt}V_C(t) = I(t) \end{gathered} \tag{1}$$

Substituting and differentiating by time again gives the following differential equation for the dynamics of the current $I(t)$ due to the stimulus $V_{in}(t)$:

$$C\tfrac{d}{dt}V_{in}(t) = LC\tfrac{d^2}{dt^2}I(t) + RC\tfrac{d}{dt}I(t) + I(t) \equiv \left(LC\tfrac{d^2}{dt^2} + RC\tfrac{d}{dt} + 1\right)I(t) \tag{2}$$

The final expression in (2) rewrites the differential equation using the so-called ***operator notation***. Equation (2) for $I(t)$ is a second-order differential equation with real, constant, positive coefficients. It is the equation of motion of a one-dimensional damped harmonic oscillator driven by an external forcing function $F(t) = C\tfrac{d}{dt}V_{in}(t)$. You have already been introduced to this system in your introductory physics lecture course and probably in other classes as well. The *driven, damped harmonic oscillator* is the archetypal resonant system, and it is one of the most important theoretical models in physics, because successful theories of much more complicated physical systems are often developed by first attempting to model them as an assemblage of such oscillators (or their quantum mechanical counterparts).

We may attack the problem of solving the differential equation (2) by first transforming it into an equivalent algebraic representation in the ***frequency domain***, using the results of Appendix A: we can take the Fourier transform of the differential equation, or, equivalently, we can just use Ohm's law and the series combination of the ***impedances*** of the R, L, and C as functions of angular frequency, ω. As indicated in Appendix A, the equivalent impedance of a series circuit is simply the sum of the individual component impedances, and each (often complex-valued) impedance is defined as the ratio of the complex-valued voltage and current ***phasors*** $V(\omega)$ and $I(\omega)$ across the component or circuit.

Note our conventions: if a voltage or current argument is frequency ω, then we are talking of a complex-valued phasor; we also use the symbol $j \equiv \sqrt{-1}$, and the convention that the complex amplitude of a sinusoid goes as $\exp(j\omega t)$.

The total impedance of the series *RLC* circuit relates $V_{in}(\omega)$ and $I(\omega)$:

$$V_{in}(\omega) = Z(\omega)I(\omega) = \left(R + j\omega L + \frac{1}{j\omega C} \right) I(\omega)\,; \quad j \equiv \sqrt{-1} \tag{3}$$

This complex-valued algebraic equation (3) provides a completely equivalent statement of the dynamical theory relating V_{in} and I to that provided by the differential equation (2). Let's examine some consequences of this theory. If we use the fact that $1/j = -j$, then we see that at one particular frequency $\omega_0 L = 1/(\omega_0 C)$ the imaginary part of the total *RLC* impedance Z in (3) vanishes, leaving only R. At this frequency the magnitude of Z is minimized, so the current I is maximized at ω_0 for a given input voltage magnitude. This special angular frequency ω_0 must satisfy the condition described by equation (4):

$$j\left(\omega_0 L - \frac{1}{\omega_0 C} \right) = 0 \;\rightarrow\; \omega_0 L = \frac{1}{\omega_0 C} \equiv Z_0 \;\rightarrow\; \boxed{\omega_0 = \frac{1}{\sqrt{LC}} \;;\; Z_0 = \sqrt{\frac{L}{C}}} \tag{4}$$

The exact cancellation of the L and C impedances at a particular frequency ω_0 is an example of ***resonance***. The first equation in (4) establishes the *resonant condition*, which in turn defines the ***resonant frequency*** (or ***characteristic frequency***) ω_0 and the ***characteristic impedance*** Z_0 for the particular *LC* pair in the circuit.

If R is much smaller than Z_0, then at resonance the impedance of the circuit is much lower (and the current through the circuit much higher) than it is at nearby frequencies. The ratio of Z_0 and R is therefore an important measure of the behavior of the system near resonance. This ***dimensionless parameter*** is called ***Q***, or the *Quality Factor* of the resonance, equation (5).

$$\boxed{Q = \frac{Z_0}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}} \tag{5}$$

Note that this is the only dimensionless quantity that can be formed from the dimensional parameters R, L, and C (other than functions of Q itself). Identifying the relevant dimensionless parameters in a theory of a physical system is extremely important; look at *Appendix C* on page 28 for a brief discussion of this topic. We can use equations (4) and (5) to express R, L, and C in terms of Q, ω_0, and Z_0:

$$\boxed{R = \frac{Z_0}{Q} \quad ; \quad L = \frac{Z_0}{\omega_0} \quad ; \quad C = \frac{1}{\omega_0 Z_0}} \tag{6}$$

If we substitute the expressions (6) into the theoretical model represented by equations (2) and (3), we get generalized expressions which may be more widely applicable than just our series *RLC* circuit. Clearly, ω_0 sets the scale for time and frequency and Z_0 sets the scale for impedance (which converts the units of the response (current) to be consistent with those of the stimulus (voltage)). If we change our scales of measurement so that $\omega_0 \equiv 1$ and $Z_0 \equiv 1$, we get generic expressions (7) and (8). The frequency domain expression (8) has an especially pleasing symmetry.

$$\boxed{\left[\frac{d^2}{dt^2} + \frac{1}{Q}\frac{d}{dt} + 1\right] y(t) = \frac{d}{dt} f(t)} \tag{7}$$

where: $t \leftarrow \omega_0 t;\quad y(t) \equiv Z_0 I(t);\quad f(t) \equiv V_{in}(t)$

$$\boxed{\left[\frac{1}{Q} + j\left(\omega - \frac{1}{\omega}\right)\right] y(\omega) = f(\omega)} \tag{8}$$

where: $\omega \leftarrow \omega/\omega_0;\quad y(\omega) \equiv Z_0 I(\omega);\quad f(\omega) \equiv V_{in}(\omega)$

With scale factors removed, Q becomes the theory's only relevant parameter which, along with the structural form of equation (7) or (8), describes the physics of the system. Your task will be to evaluate the adequacy of this theory to describe the observed behavior of your circuit.

RESPONSE TO A SINUSOID INPUT

If $V_{in}(t)$ is a sinusoid at angular frequency ω so that $V_{in}(t) = \mathrm{Re}\left[V(\omega)e^{j\omega t}\right]$, then the frequency-domain expression (3) provides the appropriate representation of the theory to use. For your experiment, however, you won't measure the current through the circuit but rather the voltage across the circuit's capacitor: $V_{out}(\omega) = V_C(\omega) = Z_C(\omega)I(\omega) = I(\omega)/j\omega C$, as shown in Figure 3 on page 3. Using the substitutions for *R*, *L*, and *C* in (6), we get a frequency-domain expression for the system's complex-valued ***gain*** (or ***transfer function***) predicted by the theory:

$$\boxed{G(\omega) \;\equiv\; \frac{V_{out}(\omega)}{V_{in}(\omega)} \;\equiv\; \left(\left(1-\frac{\omega^2}{\omega_0^2}\right) + j\frac{1}{Q}\frac{\omega}{\omega_0}\right)^{-1}} \tag{9}$$

The frequency-domain gain *G*(ω) in (9) is complex-valued. Its magnitude $|G(\omega)|$ and phase $\phi_G(\omega)$ are:

$$\boxed{|G(\omega)| = \left(\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \frac{1}{Q^2}\frac{\omega^2}{\omega_0^2}\right)^{-1/2}} \tag{10}$$

$$\boxed{\phi_G(\omega) \;=\; -\tan^{-1}\left(\frac{(1/Q)\,\omega/\omega_0}{1-\omega^2/\omega_0^2}\right) \;=\; -\left[\frac{\pi}{2} + \tan^{-1}\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)\right]} \tag{11}$$

A few words about the phase expression (11): from equation (9) it should be clear that the imaginary part of $G(\omega)$ is negative, because it is the reciprocal of an expression with a positive imaginary part. Thus $-\pi < \phi_G < 0$. The first arctangent expression in (11) shows this explicitly as the negative of the ratio of the imaginary and real parts of $1/G(\omega)$ from (9). Unfortunately, the arctangent's argument is discontinuous at the resonant frequency, $\omega = \omega_0$. The arctangent's argument in the rightmost expression in (11), on the other hand, is continuous for all $\omega > 0$, and this arctangent expression returns an angle in the branch $-\pi/2 < \theta < \pi/2$ (±90°), which is also typically the range returned by computer function calls.

Note from these expressions (10) and (11) that $|G(\omega_0)| = Q$ and $\phi_G(\omega_0) = -90°$. Plots of $G(\omega)$ and $\phi_G(\omega)$ for $Q = 20$ are shown in Figure 4 on page 7. When $Q^2 \gg 1$ the maximum of $|G(\omega)|$ occurs very close to ω_0 (see prelab problem 2); elsewhere the gain is generally ≤ 1. The phase changes very rapidly with ω near ω_0 as well. This is an example of *resonant behavior* because this rapid change in the gain near ω_0 is quite different from the system's behavior at frequencies more than a few times ω_0/Q away from ω_0, which for large *Q* can be a very small range in frequency.

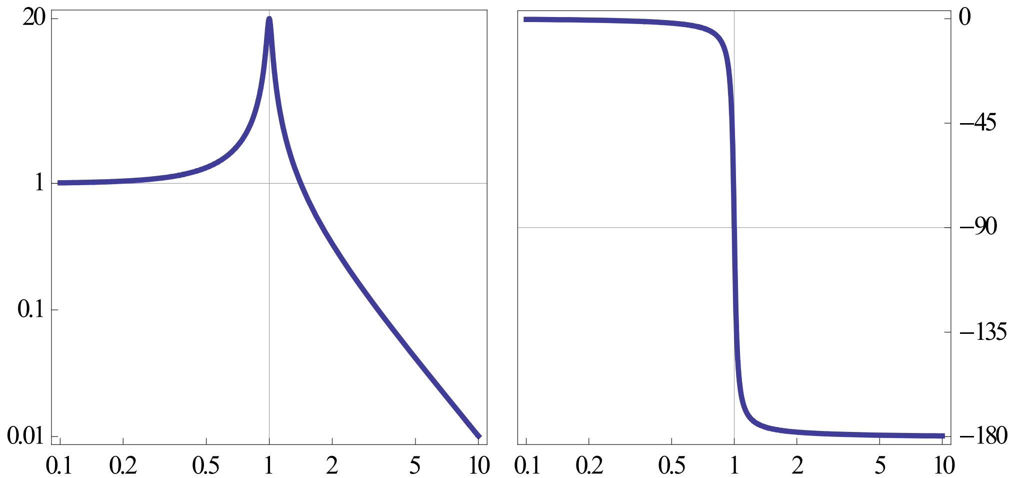


Figure 4: Magnitude and phase (in degrees) of $G(\omega)$ vs. ω/ω_0 for the series *RLC* circuit of figure 1 with Q = 20. At frequencies much lower than the resonant frequency the gain approaches 1. At frequencies much higher than the resonant frequency the gain approaches $-(\omega_0/\omega)^2$. At resonance the gain is $-jQ$. Most of the change from the low-frequency to the high-frequency asymptotic behavior occurs within a range of only a few times $1/Q$ of the resonant frequency (note the log scales for frequency and gain).

$$\boxed{\text{Resonant width: } \gamma \equiv \frac{\omega_0}{Q}} \tag{12}$$

When $Q^2 \gg 1$, γ as defined in (12) is the change in angular frequency (radians/second) across the resonance where $|G(\omega_0 \pm \gamma/2)| = |G(\omega_0)|/\sqrt{2}$ and $\phi_G(\omega_0 \pm \gamma/2) = \phi_G(\omega_0) \pm (-\pi/4)$ (or 45°). When we use frequencies f measured in hertz, we'll refer to $\gamma_f = f_0/Q$.

IMPULSE (TRANSIENT) RESPONSE

If $Q^2 \gg 1$ and $V_{in}(t)$ consists a short-lived, large-amplitude change and is then held constant, the theory predicts that the system's response will be to oscillate (or "ring", like a bell) at a frequency very near ω_0 for a number of order Q cycles before the response amplitude has decayed considerably. Let's derive this result from the differential equation (7) with the scale factors ω_0 and Z_0 put back in:

$$\left[\frac{d^2}{dt^2} + \frac{\omega_0}{Q}\frac{d}{dt} + \omega_0^2\right] Z_0 I(t) = \omega_0 \frac{d}{dt} V_{in}(t) \tag{13}$$

If $V_{in}(t)$ has been constant for a very long time, then the right-hand side of (13) has been 0 for a very long time as well, and eventually all the derivatives of $I(t)$ on the left-hand side will have vanished. This obviously implies that for constant V_{in}, $I(t \to \infty) = 0$. The solution to the now homogeneous differential equation (13) with its right-hand side set to 0 is the equation's so-called "complementary solution." In this case $I(t)$ and its derivatives are all proportional to each other, so assume a solution of the form:

$$I(t) = \mathrm{Re}\left[I_0 \, e^{-t/\tau} e^{j\omega_T t} \right] \tag{14}$$

Expression (14) describes a damped oscillation at angular frequency ω_T with a decay time constant τ. I_0 is the oscillation's complex-valued phasor at $t = 0$ and determines the initial amplitude and phase of the oscillation. Substituting this form for $I(t)$ into the homogeneous form of (13) leads to the *characteristic equation* for ω_T and τ:

$$\left(j\omega_T - \tau^{-1} \right)^2 + \frac{\omega_0}{Q}\left(j\omega_T - \tau^{-1} \right) + \omega_0^2 = 0 \tag{15}$$

The real and imaginary parts of (15) must both vanish, giving solutions for ω_T and τ:

$$\boxed{\begin{aligned} \omega_T &= \omega_0 \sqrt{1 - \frac{1}{4Q^2}} \\ \tau &= 2\left(Q/\omega_0 \right) = 2/\gamma \end{aligned}} \tag{16}$$

To determine I_0 in the expression (14), assume that $V_{in}(t)$ and its derivatives vanish everywhere except for an infinitesimal region about $t = 0$. Further assume that the time integral of $V_{in}(t)$ does not vanish within this region, but has the finite value $V_0 T_0$ so that $V_{in}(t)$ applies a short impulse to the circuit at $t = 0$ but is otherwise zero. Mathematically, $V_{in}(t) = (V_0 T_0)\,\delta(t)$, where $\delta(t)$ is the *Dirac delta function* (see this experiment's *Appendix B* on page 26). Before this impulse occurs, assume that initially $I(t)$ and all its derivatives vanish. With these conditions on $I(t)$ and $V_{in}(t)$ we can integrate equation (13) twice to determine the current and its time derivative at the instant just following the impulse. The integration of equation (13) is presented in Appendix B with results:

$$\begin{aligned} I(0+) &= V_0 T_0 \omega_0 / Z_0 \\ \tfrac{d}{dt} I(0+) &= -\gamma I(0+) \\ \text{and } I_0 &= \left(1 + j\frac{1}{\tau\,\omega_T} \right) I(0+) = \left(1 + j\frac{1}{\sqrt{4Q^2 - 1}} \right) I(0+) \end{aligned} \tag{17}$$

The notation "$I(0+)$" means the limit of $I(t)$ as $t \to 0$ from above, that is, the value of I immediately following the V_{in} impulse at $t = 0$ (and similarly for $\tfrac{d}{dt} I(0+)$).

Finally, the solution (14) for $I(t)$ becomes:

$$\boxed{\begin{aligned} I(t>0) &= \frac{V_0 T_0 \omega_0}{Z_0}\left(\cos(\omega_T t) - \frac{1}{\sqrt{4Q^2-1}}\sin(\omega_T t)\right)e^{-t/\tau} \\ \omega_T &= \omega_0\sqrt{1-\tfrac{1}{4Q^2}} \qquad \tau = 2Q/\omega_0 \end{aligned}} \tag{18}$$

The formula for the transient response time constant τ is proportional to Q and, in fact, equals $2/\gamma$ (in radians/sec). Also note that Q need not be much greater than 1 for the transient oscillation frequency ω_T to be very nearly equal to ω_0. Because $I(t) = C\frac{d}{dt}V_{out}(t)$, we can integrate equation (14) using solution (17) to get a similar result for the output voltage, equations (19). Figure 5 illustrates the resultant output current and voltage responses for $Q = 20$.

$$\boxed{\begin{aligned} V_{out}(t>0) &= \frac{V_0 T_0 \omega_0}{\sqrt{1-\tfrac{1}{4Q^2}}}\sin(\omega_T t)\,e^{-t/\tau} \\ \omega_T &= \omega_0\sqrt{1-\tfrac{1}{4Q^2}} \qquad \tau = 2Q/\omega_0 \end{aligned}} \tag{19}$$

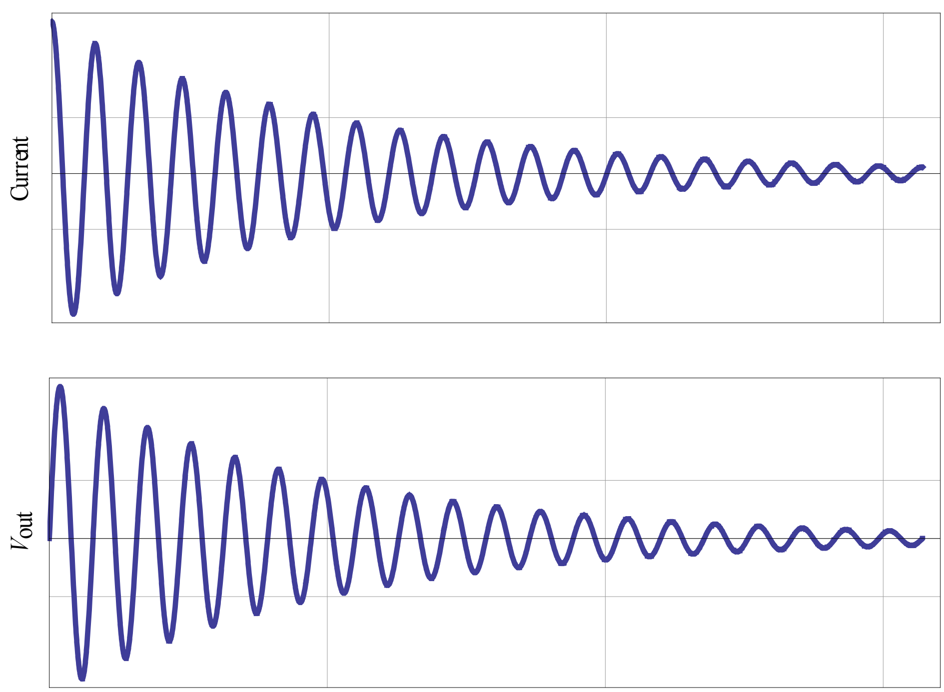


Figure 5: Response to a voltage impulse of the series *RLC* circuit of figure 1 with Q = 20. The current through the circuit and the voltage at the output are plotted as functions of time. The vertical grid lines are drawn at intervals of Q/π cycles following the impulse. The horizontal grid lines are drawn at $\pm 1/e$ of the initial amplitude. A total of Q cycles are shown, at which time the response has decayed by more than 95%. Note the phase difference between the two plots.

PROCEDURE

General approach and requirements

In solving *Prelab Problem* 3 you calculate ω_0, Z_0, Q, and $f_0 = \omega_0/(2\pi)$ using assumed, nominal values for the circuit elements. Your lab TA or the lab instructor can tell you the nominal inductance L and resistance R actually present in your set-up (either $10\,\mathrm{mH}$ and $25\,\Omega$ or $100\,\mathrm{mH}$ and $50\,\Omega$) so that you can estimate the values of these parameters for your set-up.

At some point during the lab period you should also directly measure the circuit elements' R, L, and C values. Calculations of the circuit's expected ω_0 and Q using these values and equations (4) and (5) may then be compared to your measured circuit parameters.

You will first evaluate the series *RLC* circuit's frequency-domain behavior not only in detail near its resonant frequency but also over a wide range of frequencies extending well over an order of magnitude away from the resonant frequency. As you collect data you will determine initial estimates of the circuit's resonant frequency and Q, and will also look for behavior which may prove to be inconsistent with the predictions of the simple, damped harmonic oscillator theory.

You will then reconfigure the apparatus to collect detailed data of the circuit's transient (time-domain) response following a narrow voltage pulse input. Initial in-lab estimates of the circuit's transient oscillation frequency and decay time constant give results which can be compared with those estimated from your frequency-domain observations.

The experiment set-up includes a research-grade computer-controlled data acquisition system (DAQ) which can take large amounts of very precise data. A secondary but important objective of the lab exercise is for you to become familiar with the instrumentation and software, because these things are used in several of the other experiments available in Physics 6 and Physics 7.

Frequency response measurements

Check and complete the set-up of the apparatus to first make frequency response measurements (Figure 6 and Figure 7 on page 11, and the photo, Figure 2 on page 2). Turn on the signal generator and the oscilloscope—they will each take several second to boot up and begin working. Both instruments are connected via USB to the set-up's workstation computer. The DAQ consists of special hardware installed in the computer along with an external interface box for connection to the experiment apparatus. Your instructor should give you a brief demonstration of the operation of the equipment and software you will use.

Once the apparatus is properly configured and the signal generator has finished its startup, launch the *Frequency Response* application (*FR app*) on the workstation (its icon is shown at left). After initializing, its window should display status and configuration information about the DAQ and signal generator hardware. The three tab selectors near the top of the application's display window are used to select its operating mode. Start with the ***Manual***

Frequency Response Setup

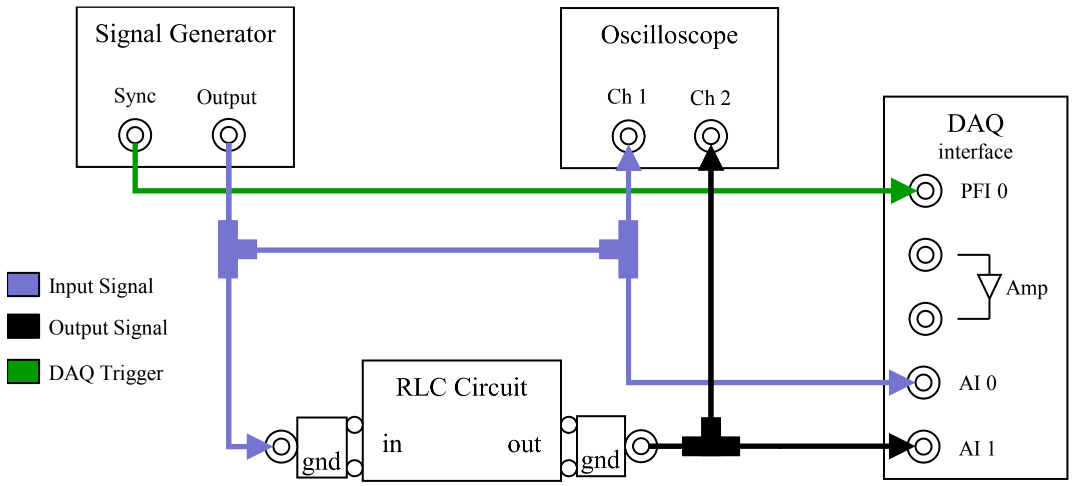


Figure 6: Frequency response general arrangement. The heavy arrows represent signal flows through BNC cables connecting the various assemblies. The BNC cables are connected to the *RLC* circuit using special adapters (see figure 7). An oscilloscope should be connected to the circuit's input and output as well to monitor the signals. The signal generator should be set to output a sine waveform (starting the *Frequency Response* application will set up the signal generator properly).

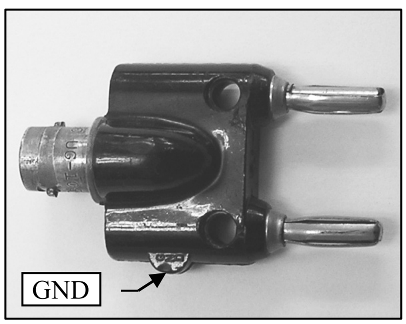


Figure 7: A BNC-Banana plug adapter. The "GND" tab on the adapter (lower center in the photo) indicates which banana plug is connected to the outer (shield) conductor of the BNC connector. Most of the lab's electronic equipment connect the outer conductors of their BNC connectors to Earth ground (through the ground pins on their AC power cords) so all of these outer conductors are connected together via this ground path. The potential of this set of conductors is used as the reference potential for voltage measurements.

Control tab. Always use this mode to ensure that your input sensitivities and data sampling parameters are set for accurate data acquisition and to perform quick checks and point estimates of the circuit's parameter values. Make sure you understand how the program is analyzing the data to produce these results. Now you're ready to collect some preliminary data.

Preliminary checks and parameter estimates

The theory of the simple, damped harmonic oscillator predicts that:

1. $\phi_G(f_0) = -90°$ and $|G(f_0)| = Q$. We use frequencies measured in Hertz.
2. $\gamma_f = f_{135} - f_{45}$, where $\phi_G(f_{135}) = -135°$ and $\phi_G(f_{45}) = -45°$ (if $Q^2 \gg 1$). It should also be the case that $|G(f_{135})| = |G(f_{45})| = |G(f_0)|/\sqrt{2}$ (if $Q^2 \gg 1$).
3. For $f \ll f_0$, $|G(f)| \to 1$ and $\phi_G(f) \to 0$.
4. For $f \gg f_0$, $|G(f)| \to (f_0/f)^2$, and the signals should be out of phase $(\phi_G = -180°)$.

The above facts provide tests to quickly and roughly check the validity of the theory for the *RLC* circuit and determine point estimates of the circuit's f_0 and Q.

Quickly find the circuit's resonant frequency f_0 by first entering the frequency you calculated in the *Prelab Problems* into the signal generator using its front-panel keypad. Consider the theory fact (1) above. Now monitor the measured phase ϕ_G using the *FR app*'s ***Manual*** mode display as you adjust the signal generator's frequency control knob to find the actual circuit f_0. If necessary, adjust the signal generator output amplitude to make sure that the measured V_{out} response peak amplitude remains below 2 volts, even at resonance. The application's measurements will now provide quick estimated values for f_0 and Q. Adjust the signal generator to various frequencies so as to roughly check the validity of the other facts (2) through (4) above.

Keep good in-lab notes of your findings!

Comprehensive data collection

Use the ***Frequency Sweep*** tab of the program to acquire accurate spectra of the frequency response of the system. Your sweep data should include fine detail of the resonant peak along with a wide frequency range showing the circuit's asymptotic behavior well away from resonance. You will probably want to complete your lab work with several sweep data files exploring various regions of the parameter space.

Save your data often!

Save your frequency sweep data by pushing the ***Save Latest*** button on the sweep display. Then **always select** the ***Save All*** radio button in the Save dialog box. This option saves all measured data regarding the sweep – amplitudes, phases, etc. You can then reload the saved data file back into the *Frequency Response* application. Data saved this way may then be imported to ***CurveFit*** using its *Data I/O* function ***Load Frequency Response Data***. A CurveFit dialog box will then open asking which aspect of the data set should be imported: gain magnitude, phase, or whatever.

Frequency sweep data sets are usually best obtained in increments. Perform an initial sweep containing relatively few data points to cover a portion of the parameter space. This data is then evaluated to determine if the DAQ input and response sensitivities are appropriate (these are set using the *Manual Control* tab) and that the signal generator output amplitude is appropriate (this must be set using the signal generator's input panel). Once you are satisfied that you can acquire accurate data, more detailed sweeps of various parts of the parameter space are performed and then merged using the sweep window controls to build up a complete data set.

For example, the signal generator amplitude must not be set so high that the response amplitude is excessive as you sweep through the resonance frequency. On the other hand, the circuit response at frequencies well above resonance will be very small, and noise will limit your data accuracy. Increasing the signal generator output amplitude for sweeps in this range can greatly improve your data.

The sweep display has a blue data cursor you can drag to individual data points using your mouse cursor. Drag the data cursor to any data point of interest. You can then:

1. Read off the point's frequency and data value in the display's cursor legend box.

2. Copy the point's frequency into the sweep ***Start Freq*** or ***End Freq*** box using the desired box's blue button.

3. Automatically change the signal generator frequency to the point's value by returning to the ***Manual Control*** mode tab, and then ensure that the acquired signal data fits appropriately into the input and response windows.

This latter functionality is very useful to check the validity of a possibly suspect data point.

More things to keep in mind

1. Make sure that you acquire detailed, high resolution sweep data around the resonance peak. Use linear gain and frequency scales for this effort to ensure that you get plenty of data points within a few Q ths of the resonant frequency.

2. As you acquire data examine the phase data plot as well as the gain magnitude plot. Also check the input amplitude and response amplitude plots. Note that the input amplitude decreases sharply near the resonance. Could the signal generator's output impedance of $50\,\Omega$ be a cause of this behavior?

3. Sweep the circuit's high-frequency asymptotic response up to at least 300 kHz. **Use log-log plot scales when sweeping over a broad frequency range**. Your data should then resemble an extended version of the gain and phase plots shown in Figure 4 on page 7. Getting good data at high frequencies will require you to make adjustments to the signal generator amplitude and the DAQ sensitivities.

4. You should check your data files using *CurveFit* to ensure that you have properly saved the data you need. Perform some preliminary fits before leaving the lab.

Linearity of the response

The damped harmonic oscillator theory is linear. This implies that $G(\omega)$, and, in particular, the system's resonance f_0 and Q, should be independent of the input signal magnitude $|V_{in}|$.

Return to the *FR app*'s ***Manual*** mode. With the signal generator's amplitude set to 100 mV peak-peak, set the generator to the system's resonant frequency. Now start to increase the signal generator output amplitude in 100 mV increments, while adjusting the DAQ input and response settings to keep acquiring accurate data.

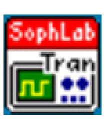
Does the system's measured gain magnitude and phase stay constant as the input is increased? Stop when the response amplitude is near the DAQ limit of 10 V. If necessary, readjust the signal generator frequency until the DAQ measured phase difference is again at −90° and record the new frequency and gain magnitude. If necessary, adjust the signal generator amplitude slightly to ensure that the acquired response data still fits in the DAQ windows so that your data are accurate.

Acquire a good sweep data set of this new resonance to compare with the lower amplitude version you already have.

Transient response measurements

A buffer amplifier installed in the DAQ interface box is used the drive the *RLC* circuit for transient response measurements. The reason for this is that our analysis of the transient behavior assumes that the input voltage to the circuit is held constant (at 0 Volts) while the circuit rings down. As illustrated in Figure 5 on page 9, the current through the circuit oscillates as the circuit rings, so the source driving the circuit must have a very low output impedance — otherwise the source voltage will vary in response to this oscillating current. The buffer amplifier's output impedance is less than an ohm at the circuit's resonant frequency, so it maintains an output voltage which is very nearly independent of the current through the circuit.

Reconfiguring the experiment apparatus for transient response measurements requires only one more BNC cable: first disconnect the existing cable from the signal generator output and reconnect it to the buffer amplifier output on the DAQ interface box. Use the additional cable to connect the signal generator output to the buffer amplifier's input. Figure 8 on page 15 illustrates these changes.



Exit the *Frequency Response* application. Set the signal generator output amplitude to 2 V peak-peak and set its frequency to 1 kHz. Now start the *Transient Response* application (its icon is shown at left). After initializing, the application opens two windows. The top window allows adjustments to DAQ settings which will be unnecessary for

Transient Response Setup

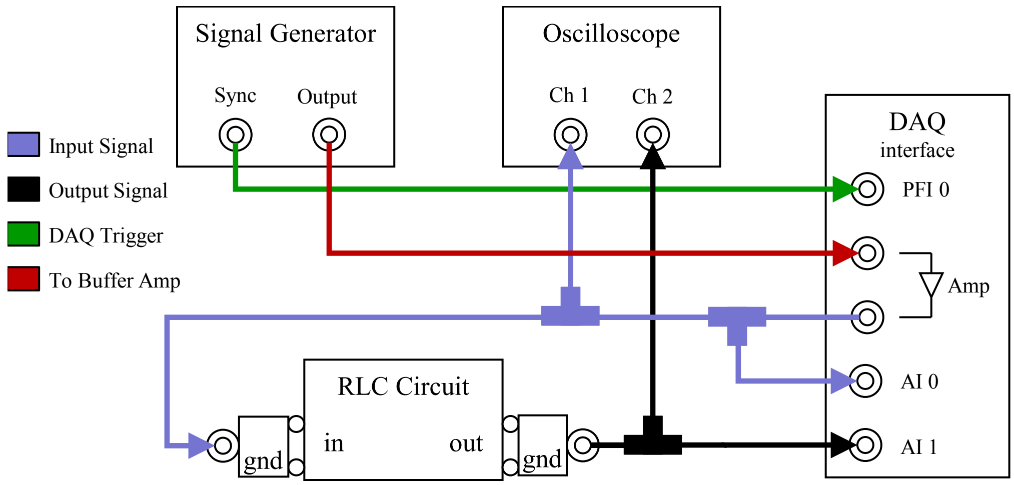


Figure 8: Transient response general arrangement. The ×1 *buffer amplifier* in the DAQ interface box isolates the *RLC* circuit from the signal generator so that its 50 Ω output impedance does not cause a variation in the *RLC* input voltage as the circuit rings down. Note that the only changes to the frequency response arrangement (Figure 6) are to (1) remove the input signal connection from the signal generator output and reconnect it to the DAQ amplifier output (blue connection), and (2) add a BNC cable connection between the signal generator output and the DAQ amplifier input (red connection).

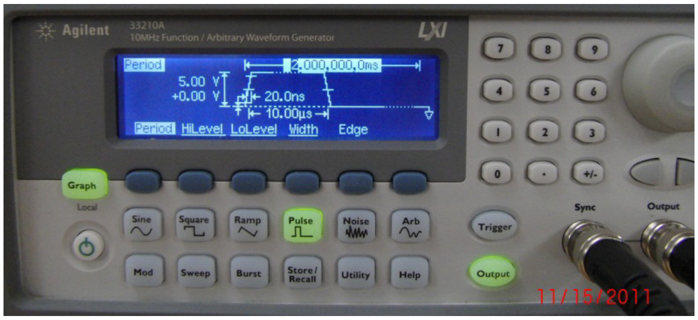
this experiment, so you can close this window. The main window appears similar to the front panel of an oscilloscope. To activate data acquisition, push the green **RESUME** button in the lower right area of the application window. You should then see the input signal as a yellow trace and the response as a blue-green trace. At this frequency they should have similar amplitudes and phases. Play a bit with the window's controls to familiarize yourself with their operation.

You should first exit the *Frequency Response* application whenever using *Transient Response*, and vice versa. Otherwise the two applications can generate conflicting commands to the DAQ drivers and hardware.

Configuring the signal generator

Refer to equations (19) and Figure 5 on page 9, which describe the theory's prediction of the circuit's response to a narrow pulse input. Your task will be to excite the circuit with a reasonably narrow, positive-going voltage pulse and then hold the input voltage at zero. The *Transient Response* application (*TR* app) can then record $V_{out}(t)$ data while the response signal rings down.

The interval between successive excitation pulses should be long enough for the output signal to decay to a very small value. Figure 5 displays only Q cycles of the ringing, and from this plot it is clear that the excitation pulse interval should be at least $2Q$ to $3Q$ cycles. For your system a pulse interval of about $200/f_0$ should be more than adequate. Because your circuit's $Q^2 \gg 1$,



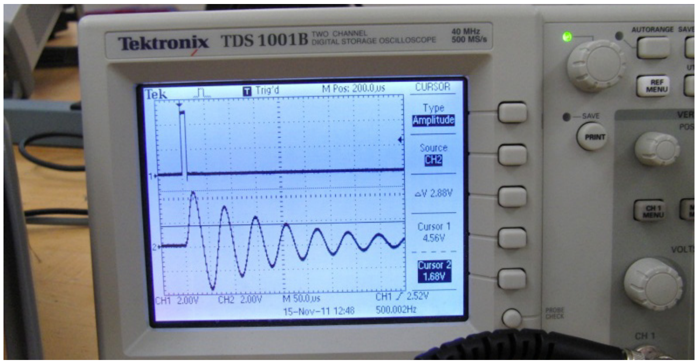


Figure 9: Photos showing a typical signal generator pulse output configuration for transient response measurements and an oscilloscope display of a pulse and a circuit's response to it. The signal generator display is obtained by selecting the generator's *Graph* button as shown. When properly configured, the Transient Response application display will be similar to the oscilloscope's but it can acquire much more accurate response waveform data.

the initial ringing amplitude should be very close to equation (19)'s $V_0 T_0 \omega_0$, and the ringing frequency $f_T = \omega_T / 2\pi$ should be very close to the system's resonant frequency f_0.

First select the signal generator's **Pulse** waveform button. Next use the **Period** menu button to set the pulse interval to an appropriate value using the criteria outlined in the previous paragraph. The pulse **HiLevel** (which will represent V_0 in equation (19)) should be set to a volt or two in order to obtain low-noise data, and the pulse **LoLevel** should be set to 0. The pulse **Width** then equals T_0 in equation (19). Why not pick a width so that $T_0 \omega_0 = 1$ (as in the example shown in Figure 9)? With this choice the pulse will be sufficiently narrow, and the initial ringing amplitude should be very close to the pulse **HiLevel** value (V_0).

Transient Response application notes

For accurate measurements of the decay you should acquire ~Q cycles of the circuit's transient response. Set the *TR app* horizontal **TIME** (in the app's blue control area) to a value sufficiently large to capture this many cycles. The application's default **Number of Points** will be way too small to resolve this many cycles, so change that setting to 1000 or so. The display area should now show something recognizable as a decaying oscillation. You may select the **CH1 OFF** button so that the app's **CH2** data refreshes more frequently. Make further adjustments as necessary to obtain a good display of the transient response oscillating decay.

Quick estimation of *Q* from the decay

From your answer to Prelab problem 4 you know that N_e, the number of cycles required for the response to decay by a factor of *e* (to about one third), when multiplied by π should equal the system Q. If it is easier to estimate τ, the time required to "*e*-fold," rather than count cycles, then a little reflection should reveal that $N_e = \tau \times f_0$, so it is also true that $Q = \pi \, \tau \, f_0$. Now use the *TR app* display to estimate Q. How does this estimate compare to the frequency response estimates?

Saving the transient response data

In order to generate uncertainties in the amplitude data, the measurements must be repeated. Accomplish this using the TR app's **Sweep Averaging** controls. Averaging 5 sweeps should be sufficient. Once the data look good, **PAUSE** the data acquisition and **SAVE** the data. Make sure you save the **CH2** data!

> When saving data acquired with the ***Transient Response*** application, **make sure you save the channel you want** (**CH 2** has the circuit's ringing response to the pulse, but **CH 1** is the default).
>
> The data is saved in the ***Standard CurveFit*** file format.

Load the saved data into *CurveFit* to check that it was saved properly. This data may be fit using the *FitAnyFunction.nb Mathematica* notebook found in the *CurveFit* folder in your computer's *Documents* folder.

Initial ringing amplitude check

Speed up the *TR app*'s Horizontal **TIME** so that only the first couple of cycles of the ringing are shown. Reactivate **CH1** if necessary and set both channel to the same **VOLTS F/S** value. Now check to see that the initial ringing amplitude is consistent with (19)'s $V_0 T_0 \omega_0$. Remember that V_0 and T_0 are the input pulse's amplitude and width, respectively.

Before leaving the lab

> Have you checked your data files using *CurveFit* to ensure that you have the data needed for your detailed analysis?

Have you measured the actual *R*, *L*, and *C* values used in your set-up?

Quit the *TR app* and the *FR app*. Reconfigure the apparatus for frequency response measurements (Figure 6 on page 11).

Turn off the signal generator and oscilloscope.

> Do you have any final questions for your TA or the lab instructor?

ANALYSIS AND CONCLUSIONS

Frequency response data analysis

Your analysis of the frequency response data should be thoughtful and thorough. Provide a *quantitative* evaluation of how accurately the theory (as represented by equations (10) and (11) on page 6) describes the system's response to a sinusoidal stimulus at various frequencies. *CurveFit* provides fitting functions for both the magnitude and phase of the system's complex-valued transfer function vs. frequency, $G(f)$, where your measured frequency $f = \omega/2\pi$.

> Frequency response data is imported to *CurveFit* using its ***Data I/O*** function ***Load Frequency Response Data***. A *CurveFit* dialog box will then open with options to select which aspect of the data set should be imported: gain magnitude, phase, or whatever.

As you saw in your acquired data, your observed $G(f)$ at high frequencies exhibited behavior which conflicts with the simple damped harmonic oscillator theory prediction. However, the theory might nevertheless be quite accurate and useful, especially near and below the observed resonant frequency. Assuming that this is the case, the theory's parameters ω_0 and Q remain appropriate and useful. Your analysis must provide values (with uncertainties) of f_0 and Q (or, equivalently, γ_f), and it should compare the estimates of these parameters which result from fitting the magnitude and the phase data.

> Don't confuse frequencies measured in hertz (such as f_0 and γ_f) with *angular* frequencies (such as ω_0 and γ). Remember where the 2π's go!

To reiterate: when fitting the data to the theoretical model, you should perform at least two fits to the gain magnitude data and another two fits to the gain phase data: (*a*) attempt to fit only a narrow range of frequencies within a few γ of the resonance, and (*b*) attempt a fit of the entire data set, including the high-frequency response.

> Fits near resonance should use linear scales, but data plots extending far from resonance should use log frequency scales and log gain magnitude scales (as in Figure 4 on page 7).

At what level of accuracy can you claim that the theory describes the behavior of the system, especially near resonance? How consistent with each other are the model fits to the amplitude and phase data? How about the fits' f_0 and Q values? Thoughtful consideration of the fits' parameter uncertainties and χ^2 values are required to address these questions. Are your observed f_0 and Q consistent with what you calculate from your measured R, L, and C values? Assume the measured L value accurately reflects the inductance in the circuit; what must R and C be to account for your observed results? How then does your calculated R_L compare to your measured DC resistance of the inductor?

So what is going on at high frequencies? Don't forget to look at the phase response as well as the amplitude response in this regime. Do you have evidence of the presence of one or more additional resonances? How do these resonant frequencies, if any, compare to f_0? Is this behavior predicted by the theory? Can you come up with any ideas as to how to extend the theory to include these observations?

Finally, don't forget to illustrate and address the issue raised by the variation in the observed f_0 and Q as the input signal amplitude was increased. Can any linear theory model such an effect?

Transient response data analysis

Use the *FitAnyFunction.nb* notebook found in the *CurveFit* directory to fit your saved transient response data (whose format is a standard *CurveFit* data file). Carefully follow the instructions in the notebook to properly perform the fit. Are the resulting fit parameters f_0 and τ consistent with your frequency response analysis? Is the fit's initial amplitude parameter A consistent with your input pulse amplitude and width ($A \approx V_0 T_0 \omega_0$)?

Any comments about the fit residuals and the χ^2 value? What if you fit a subset of the data where the amplitudes are smaller? Any ideas as to what causes the pattern in the fit residuals?

Conclusions

Provide a brief, thoughtful summary of your findings.

PRELAB PROBLEMS

1. Derive equation (9) for the complex gain $G(\omega)$ from the expressions in (8) and the facts that $V_{out}(\omega) = V_C(\omega) = Z_C(\omega)I(\omega) = I(\omega)/j\omega C$ and $Z_0 = 1/j\omega_0 C$.

2. Use equation (10) for $|G(\omega)|$ to show that $|G(\omega_0)| = Q$, but that the maximum value of $|G(\omega)|$ is actually given by:

$$|G(\omega_{\max})| = \frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}; \text{ where } \omega_{\max} = \omega_0\sqrt{1-\frac{1}{2Q^2}}$$

3. Given: $L = 10\,\text{mH}$; $C = 10\,\text{nF}$; $R = 50\,\Omega$
   What are $\omega_0, Z_0, Q, \gamma, \tau$? What are $f_0 = \omega_0/(2\pi \text{ rad/cycle})$ and $\gamma_f = \gamma/(2\pi \text{ rad/cycle})$?
   What would R_L be if the measured $Q = 17$?
   What do f_0 and Q become if L is changed to $100\,\text{mH}$?

4. Use equations (16) to show that during the transient response ring-down the amplitude decays to $1/e$ of its initial value after Q/π cycles (periods). Assume $Q^2 \gg 1$ so that $\omega_T \approx \omega_0$.

5. Provide a brief, verbal description of your experiment procedure to your TA.

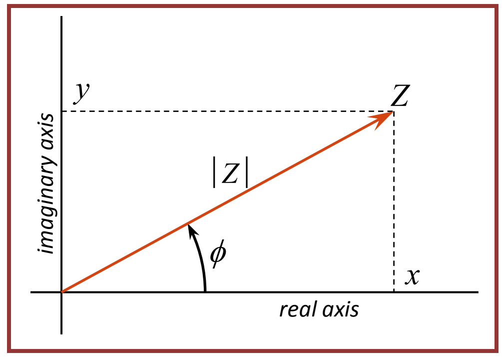
APPENDIX A: IMPEDANCE

A review of some algebra of complex numbers

Here is a terse review of the elementary mathematics of *complex numbers* which we will need to manipulate frequency-domain signals and impedances.

Given: $Z, W \in \mathbb{C}$; $x, y, \phi \in \mathbb{R}$

The complex number Z may be represented as a vector in a plane as shown at right. We then have the following Cartesian (real and imaginary parts) and polar coordinate (exponential) representations of Z:



$$Z = x + j\,y = |Z|\,e^{j\phi}$$

$$\mathrm{Re}[Z] \equiv x; \quad \mathrm{Im}[Z] \equiv y$$

$$|Z| = \sqrt{x^2 + y^2}; \quad e^{j\phi} \equiv \cos\phi + j\sin\phi; \quad x = |Z|\cos\phi; \quad y = |Z|\sin\phi; \quad \phi = \arctan(y/x)$$

$$j = e^{j\pi/2}; \ -1 = e^{j\pi}; \ -Z = -x - j\,y = |Z|\,e^{j(\phi \pm \pi)}$$

Terminology:

x: real part of Z; y: imaginary part of Z; $|Z|$: magnitude of Z; ϕ: phase of Z

Conjugates, magnitudes, reciprocals:

$$Z^* = \mathrm{conj}[Z] \equiv x - j\,y = |Z|\,e^{-j\phi}; \quad |Z|^2 = Z\,Z^* = Z^*Z; \quad |1/Z| = 1/|Z|$$

$$\mathrm{Re}[Z] = \tfrac{1}{2}(Z + Z^*); \quad \mathrm{Im}[Z] = \tfrac{1}{2j}(Z - Z^*); \quad \mathrm{Re}[Z^*] = \mathrm{Re}[Z]; \quad \mathrm{Im}[Z^*] = -\mathrm{Im}[Z]$$

$$\frac{1}{Z} = \frac{1}{x + j\,y} = \frac{x - j\,y}{x^2 + y^2} = \frac{1}{|Z|}\,e^{-j\phi}; \quad \left(\frac{1}{Z}\right)^* = \frac{1}{Z^*}; \quad \boxed{\frac{1}{j} = -j}$$

Products of two complex numbers:

$$\text{Let:} \quad Z = x_Z + j\,y_Z = |Z|\,e^{j\phi_Z}; \quad W = x_W + j\,y_W = |W|\,e^{j\phi_W}$$

$$Z\,W = |Z|\,|W|\,e^{j(\phi_Z + \phi_W)} = x_Z x_W - y_Z y_W + j\,(x_Z y_W + x_W y_Z); \quad (Z\,W)^* = Z^* W^*$$

$$|Z\,W| = |Z|\,|W|; \quad |Z/W| = |Z|/|W|$$

Linear functions and time derivatives:

If $F(Z)$ is a complex-valued function of its complex argument Z, then we can write:

$F(Z) = u(Z) + j\,v(Z)$, where u and v return real values for every Z

If $F(Z)$ is also ***linear***, then:

$F(Z) = F(x + j\,y) = F(x) + j\,F(y)$ [Note: F(real x) may be complex-valued]

Let time $t \in \mathbb{R}$, and let $Z(t) = x(t) + j\,y(t)$, where $x(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$. Then:

$$\frac{d}{dt}x(t) + j\frac{d}{dt}y(t) = \frac{d}{dt}[x(t) + j\,y(t)] = \frac{d}{dt}Z(t); \quad \boxed{\therefore \frac{d}{dt}\mathrm{Re}[Z(t)] = \mathrm{Re}\left[\frac{d}{dt}Z(t)\right]}$$

so we may exchange the order of time differentiation and taking the real part. This is a most important result.

Complex-valued representation of sinusoids

We can represent a sinusoidal function of time using complex numbers:

$$\begin{aligned} &\text{If:} \quad y(t) = y_{\max}\cos(\omega t + \phi) \\ &\text{Then:} \quad y(t) = \mathrm{Re}\left[Y\,e^{j\omega t}\right]; \ \text{where:} \ Y = y_{\max}\,e^{j\phi} \end{aligned} \tag{20}$$

The complex "amplitude" Y is called a *phasor* since it determines both the amplitude and phase of $y(t)$. If the phase ϕ of Y is greater than 0, then the phase of $y(t)$ "leads" the phase of $\cos\omega t$; if $\phi < 0$ the phase "lags" that of $\cos\omega t$. It will generally be the case that the phasor $Y = Y(\omega)$, a function of the angular frequency. It *is not the case*, however, that the phasor is a function of time t, because all the time dependence of $y(t)$ is captured in the $e^{j\omega t}$ term. Since the only factors which differentiate various sinusoidal functions all at the same frequency are the differences in their phasors, we can simply refer to the functional relationship (20) as

$$y(t) \leftrightarrow Y(\omega)\ ;\ \text{with } y(t) \in \mathbb{R}\ ,\ \text{whereas } Y(\omega) \in \mathbb{C} \tag{21}$$

What this implies is that when we consider *phasors* as specifications of sinusoidal functions of time we are using a *frequency-domain* representation of the functions, that is: we are working in the *Fourier transform space* of functions of time.

More generally, a real-valued function of time (not necessarily a sinusoid) may be represented by a complex-valued function of frequency by using a Fourier transform and it inverse (the pair given here is consistent with our sinusoid representation (20)):

$$
\begin{aligned}
x(t) &= \mathrm{Re}\left[\int_0^{\infty} X(\omega)e^{j\omega t}\,d\omega\right] \\
X(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt
\end{aligned}
\tag{22}
$$

So $x(t)$ and $X(\omega)$ in equations (22) are called a *Fourier transform pair*. Mathematicians have extensively studied just what sorts of functions can be represented this way, and what exactly is meant by the "=" in (22). We won't worry about such issues here.

Time derivatives and phasors

We can evaluate the frequency representation of the time derivative of a function $f(t)$ as follows:

$$
\begin{aligned}
f(t) &= \mathrm{Re}\left[\int_0^{\infty} F(\omega)e^{j\omega t}\,d\omega\right] = \tfrac{1}{2}\left(\int_0^{\infty} F(\omega)e^{j\omega t}\,d\omega + \int_0^{\infty} F^*(\omega)e^{-j\omega t}\,d\omega\right) \\
\frac{df}{dt} &= \tfrac{1}{2}\left(\int_0^{\infty} F(\omega)\frac{d}{dt}\left(e^{j\omega t}\right)d\omega + \int_0^{\infty} F^*(\omega)\frac{d}{dt}\left(e^{-j\omega t}\right)d\omega\right) \\
\frac{df}{dt} &= \tfrac{1}{2}\left(\int_0^{\infty} j\omega F(\omega)e^{j\omega t}\,d\omega + \int_0^{\infty} (-j\omega F^*(\omega))e^{-j\omega t}\,d\omega\right) = \mathrm{Re}\left[\int_0^{\infty}\left(j\omega F(\omega)\right)e^{j\omega t}\,d\omega\right]
\end{aligned}
$$

Time differentiation of a function $f(t)$ corresponds to multiplying its Fourier transform by $j\omega$.

$$
\boxed{
\begin{array}{lccc}
\text{If:} & f(t) & \leftrightarrow & F(\omega) \\
\text{Then:} & \dfrac{df}{dt} & \leftrightarrow & j\omega F(\omega)
\end{array}
}
\tag{23}
$$

A note about sign conventions

We have defined our sinusoid representation so that its time variation goes as $e^{j\omega t}$. This is the convention used in electrical engineering and similar disciplines. With this choice a wave propagating in the positive direction along the x-axis would have phase variation $e^{j(\omega t - kx)}$. This is not the usual sign convention chosen by physicists when considering wave propagation, who would rather assign a phase variation of $e^{i(kx-\omega t)}$. This choice would imply that the time derivative operation of expression (23) would result in a minus sign on $j\omega$. An unfortunate consequence of this alternate sign convention would be that the normal choice for the impedance representation of a capacitor or inductor (to be presented in the next section) would have the wrong sign!

You may have noticed that this lab write-up uses $j = \sqrt{-1}$, rather than the mathematicians' *i*. The reason for this choice is two-fold: (*a*) it is consistent with the notation used in electronics texts, so that it is not confused with the symbol for a current (which is often *i* in these texts); and (*b*) it emphasizes the electrical engineering sign convention choice for the phase variation with time $e^{j\omega t}$.

Extending Ohm's law

Consider voltages and currents in some circuit containing an ideal resistor with resistance *R*. Ohm's law relates the resistor's time-varying voltage and current $v_R(t) = R\, i_R(t)$, and, since *R* is a real number, a corresponding relation exists between the phasors in frequency space:

$$\text{Resistor:}\quad \begin{cases} v(t) = R\, i(t) \\ V(\omega) = R\, I(\omega) \end{cases} \tag{24}$$

where the voltage and current phasors in (24) have the same phase because $R > 0$. This is Ohm's law for a resistance *R* in the frequency domain. For an inductor or capacitor, on the other hand, we can use the relation (23) to obtain:

$$\text{Inductor:}\quad \begin{cases} v(t) = L \tfrac{d}{dt} i(t) \\ V(\omega) = \left(j\omega L \right) I(\omega) \end{cases} \tag{25}$$

$$\text{Capacitor:}\quad \begin{cases} C \tfrac{d}{dt} v(t) = i(t) \\ \left(j\omega C \right) V(\omega) = I(\omega) \end{cases} \tag{26}$$

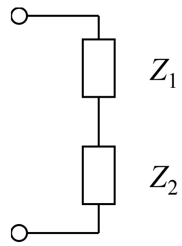
The phasor equations in (25) and (26) look like Ohm's Law except that the factors relating the voltage and current are no longer real and are also functions of the frequency. Nevertheless, we can generalize Ohm's law to include these more general phasor relationships by introducing the concept of the ***complex-valued impedance***, *Z*:

$$\boxed{\begin{array}{c} V(\omega) = Z(\omega)\, I(\omega) \\ Z_R = R; \quad Z_L = j\omega L; \quad Z_C = \dfrac{1}{j\omega C} \end{array}} \tag{27}$$

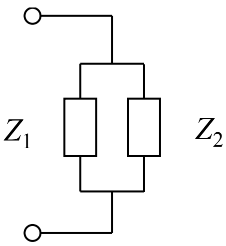
The reciprocal of the impedance is called the ***admittance***, $Y(\omega) \equiv 1/Z(\omega)$.

Series and parallel combinations and the voltage divider

Series and parallel circuit connections of impedances follow the same rules as for resistors:



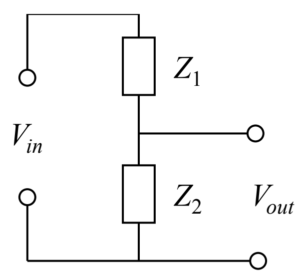
$$Z = Z_1 + Z_2$$



$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

Series impedances (and voltages) add.
Parallel admittances (and currents) add.

Voltage divider:



$$\boxed{V_{out} = \frac{Z_2}{Z_1 + Z_2} V_{in}}$$

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{|Z_2|}{|Z_1 + Z_2|}$$

$$Phase\left(\frac{V_{out}}{V_{in}}\right) = Phase(V_{out}) - Phase(V_{in})$$

APPENDIX B: DERIVATION OF THE IMPULSE RESPONSE

The Dirac delta function

The Dirac delta function, $\delta(t)$, will be used as a theoretical idealization of a very short, impulsive excitation of the series *RLC* resonant system. The defining properties of the delta function are:

$$\text{If} \quad t \neq 0, \quad \text{then} \quad \delta(t) = 0, \quad \therefore \lim_{t \to 0} \delta(t) = 0$$
$$\int_R f(t)\delta(t)\,dt = \begin{cases} f(0), & \text{If the region } R \text{ includes the origin} \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

and $f(t)$ is any well-behaved function in R.

So $\delta(t)$ must be an infinitely narrow, infinitely high "spike," or impulse which has a finite, nonzero integral. Unfortunately, such a function can't exist using the ordinary definition of a function, but it can be defined rigorously as a special limit of a sequence of ever-narrower functions (which the mathematicians call a *distribution*).

Integrating the differential equation

We can use the delta function to represent an idealized impulsive stimulus to the input of the resonant system:

$$\begin{aligned} &V_{in}(t) = V_0 T_0\, \delta(t), \text{ so that:} \\ &V_{in}(t \neq 0) = 0; \quad \lim_{\varepsilon \to 0+} \left(\int_{-\varepsilon}^{\varepsilon} V_{in}(t)\,dt \right) = V_0 T_0 \end{aligned} \tag{29}$$

where "$\varepsilon \to 0+$" means that ε approaches 0 through positive values. Integrating equation (13) to get rid of the derivative of $V_{in}(t)$ and then substituting (29) gives the integro-differential equation:

$$L \frac{d}{dt} I(t) + R\, I(t) + \frac{1}{C} \int_{-\infty}^{t} I(t')dt' = V_0 T_0\, \delta(t) \tag{30}$$

where we've used the fact that $I(t)$ and its derivatives vanish for $t < 0$. To proceed further we must assume that $I(t)$ is integrable in the ordinary sense, even at $t = 0$, so that:

$$\begin{aligned} &\int_{-\infty}^{\varepsilon} I(t)\,dt = \int_{-\infty}^{-\varepsilon} I(t)\,dt + \int_{-\varepsilon}^{\varepsilon} I(t)\,dt = 0 + \int_{-\varepsilon}^{\varepsilon} I(t)\,dt \\ &\lim_{\varepsilon \to 0+} \left(\int_{-\varepsilon}^{\varepsilon} I(t)\,dt \right) = \lim_{\varepsilon \to 0+} 2\varepsilon\, \overline{I}(\varepsilon) = 0 \end{aligned} \tag{31}$$

$\overline{I}(\varepsilon)$ in (31) is assumed to be some finite average value of $I(t)$ for $-\varepsilon \le t \le \varepsilon$. This would certainly be true if $I(t)$ were to remain finite over the infinitesimal time interval of the impulse in

$V_{in}(t)$, but all that is required is that $\int I(t)\,dt$ over any infinitesimal interval must vanish (unlike the case for $\delta(t)$!). As we shall see, both $I(t)$ and $\frac{d}{dt}I(t)$ have step-discontinuities at $t=0$.

Given (31), consider (30) in the limit as $t \to 0+$, that is, immediately after the impulse in $V_{in}(t)$. The right-hand side and the integral on the left-hand side of (30) vanish in this case, so we get a relationship between $I(t)$ and $\frac{d}{dt}I(t)$ at the instant just after the impulse:

$$\left(L\frac{d}{dt}I(t) \;+\; R\,I(t) \;+\; \frac{1}{C}\int_{-\infty}^{t} I(t')dt' \right) = V_0T_0 \lim_{t\to 0+} \delta(t) \;=\; 0$$

$$L\frac{d}{dt}I(0+) \;+\; R\,I(0+) \;+\; 0 \;=\; 0$$

$$\therefore \boxed{\frac{d}{dt}I(0+) \;=\; -\left(R/L\right)I(0+) \;=\; -\left(\omega_0/Q\right)I(0+) \;=\; -\left(2/\tau\right)I(0+)} \qquad (32)$$

Integrate (30) again and again consider the limit as $t \to 0+$. Then:

$$\lim_{t\to 0+}\left(L\,I(t) \;+\; R\int_{-\infty}^{t} I(t')dt' \;+\; \frac{1}{C}\int_{-\infty}^{t}\int_{-\infty}^{t'} I(t'')dt''dt' \right) \;=\; \lim_{t\to 0+}\int_{-\infty}^{t} V_0T_0\,\delta(t')\,dt'$$

$$L\,I(0+) \;+\; 0 \;+\; 0 \;=\; V_0T_0$$

$$\therefore \boxed{I(0+) \;=\; V_0T_0/L \;=\; V_0T_0\omega_0/Z_0} \qquad (33)$$

Equations (32) and (33) are the results that were presented in the initial conditions (17) on page 8. To get the value for I_0 given in (17), assume that $I_0 = x + jy$, and calculate $I(0+)$ and $\frac{d}{dt}I(0+)$ from (14), keeping in mind the result (32):

$$I(t) \;=\; \mathrm{Re}\left[I_0\exp\left(j\omega_T t - t/\tau\right)\right] \;=\; \mathrm{Re}\left[(x+jy)\exp\left(j\omega_T t - t/\tau\right)\right]$$

$$\therefore \boxed{I(0+) \;=\; x}$$

$$\frac{d}{dt}I(t) \;=\; \frac{d}{dt}\mathrm{Re}\left[I_0\exp\left(j\omega_T t - t/\tau\right)\right] \;=\; \mathrm{Re}\left[(x+jy)\frac{d}{dt}\exp\left(j\omega_T t - t/\tau\right)\right]$$

$$= \mathrm{Re}\left[(x+jy)\left(j\omega_T - 1/\tau\right)\exp\left(j\omega_T t - t/\tau\right)\right]$$

$$\therefore \frac{d}{dt}I(0+) \;=\; -\left(x/\tau + y\omega_T\right) \;\to\; y \;=\; -\left(x/\tau + \frac{d}{dt}I(0+)\right)\Big/\omega_T \;=\; -\left(x/\tau - 2x/\tau\right)\Big/\omega_T$$

$$\therefore \boxed{y \;=\; x\big/\left(\omega_T\tau\right)}$$

$$\therefore \boxed{I_0 \;=\; \left(1 + j\big/\left(\omega_T\tau\right)\right)\,I(0+)} \qquad (34)$$

APPENDIX C: Q AND OTHER DIMENSIONLESS PARAMETERS

For the *RLC* circuit considered in this experiment, we identified the *dimensionless parameter Q*:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{5}$$

This is the only combination of a resistance, a capacitance, and an inductance which has no units, i.e., is a pure number (of course, functions of Q are also dimensionless, but these are not independent of Q). As explained in the text, ω_0 and Z_0 are scale factors, and setting them to 1 reveals the scale-free structure of the theory, equations (7) and (8). We repeat this scale-free expression of the theory (the dots imply time derivatives):

$$\ddot{y}(t) + \frac{1}{Q}\dot{y}(t) + y(t) = \dot{f}(t)$$

$$\left[\frac{1}{Q} + j\left(\omega - \frac{1}{\omega}\right)\right] y(\omega) = f(\omega)$$

It is not possible to choose different units for R, L, and C (while maintaining their actual physical behaviors) such that the value of Q defined in (5) changes, as long as the units you use are dimensionally correct. So this dimensionless value Q may be considered to be the only true "free" parameter which, along with the structure of the equations, determines the essential physics of any system described by this theory. Thus, ignoring scale factors may reveal similarities in the behaviors of quite distinct physical systems which are nevertheless described by a single fundamental, scale-free, theoretical structure. This potential for *unification* is a powerful motivator for theoretical efforts.

Varying the value of Q would lead to an entire family of systems related by the underlying structure of the equations. Systems with different Q values in this family may behave very differently, however, even though the structure of the equations remains the same. Engineers can tailor their inventions by manipulating the values of dimensionless parameters such as Q by making judicious choices of components or other aspects of their designs.

Here are some other examples of dimensionless parameters in physical theories:

- Γ : The *ratio of specific heats* $\Gamma = C_P / C_V$ of a dilute gas. This number is related to the number of internal degrees of freedom a gas molecule has for storing energy.
- M : The *Mach number* is the ratio of the speed of a fluid flow to the speed of sound in the fluid. The physics of fluid flow are very different for $\mathrm{M} \ll 1$, $\mathrm{M} \sim 1$ and $\mathrm{M} \gg 1$.
- α: The *Fine Structure Constant* $\alpha = e^2/\hbar c$ (Gaussian units) $\approx 1/137$. This is a fundamental constant of nature which is even more important to the behavior of quantum electrodynamics than Q is to the behavior of a resonant system. The universe would be very different if this number were very different from its observed value.