

## Experiment 16

## ANALOG FOURIER ANALYSIS

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## INTRODUCTION

In this experiment you will perform a Fourier analysis of a couple of non-sinusoidal, periodic functions (a square wave and a triangle wave) using a simple, high-Q filter. In real life the method used for such analyses may be quite different; for example, one may choose to acquire a long sequence of samples of the instantaneous amplitude of the signal and then perform a numerical Fourier transform of the data, as is done by the “single-tone analysis” routine in the *Frequency Response* program you used for Experiment 2. A commercial spectrum analyzer, however, usually works by employing a method similar to the one used in this experiment.

In this experiment you will be using a LC band-pass filter with a fixed center (resonant) frequency. A signal generator will be used to generate the periodic waveform to be analyzed; by tuning the frequency of the signal we can present various harmonics of the waveform to the LC filter. The filter’s output voltage is then precisely measured; this voltage is proportional to the amplitude of the harmonic present in the input waveform.

## THEORY

Real-valued, periodic functions important to physics may be written as the sum of an infinite number of sine and cosine terms as follows:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \quad (1)$$

where  $\omega_0$  is the angular frequency of the fundamental period (lowest frequency) of the periodic function. If  $T$  is the period of the waveform, then  $\omega_0 = 2\pi/T$  and the Fourier coefficients are given by ( $n$  ranges over the nonnegative integers):

$$a_n = \frac{2}{T} \int_T f(t) \cos(n\omega_0 t) dt \quad (2)$$

$$b_n = \frac{2}{T} \int_T f(t) \sin(n\omega_0 t) dt \quad (3)$$

The integrations are performed over a single, complete period of the waveform. This analysis is really just an extension of the idea of expressing an arbitrary vector as a linear combination of a set of basis (or “unit”) vectors, such as expressing a 3-dimensional vector in terms of the Cartesian basis set  $\{\hat{x}, \hat{y}, \hat{z}\}$ . In this case the arbitrary “vector” is the periodic function  $f(t)$ , and the basis set consists of the countably infinite set of sine and cosine harmonics. The integrals (2) and (3) are the equivalent of “dot products” of the function with the various members of the set of basis functions.

Consider again the integrals (2) and (3). They are clearly calculations of the average values of their various integrands over a period of the waveform (obviously  $a_0/2$  is just the average value of  $f(t)$ ); consequently their values are independent of the period  $T$  and depend only on the shape and overall amplitude of  $f(t)$ . So if we change the frequency of the waveform but leave its shape and amplitude constant, then the values of the coefficients  $a_n$  and  $b_n$  will also remain constant.

The method we will use to measure the various Fourier coefficient values depends on this latter fact. The high-Q filter we will use to eliminate all but a single harmonic from our waveform has a fixed center frequency ( $f_{\text{filter}}$ ) of approximately 60 kHz and a Q of about 280. If we wish to measure the amplitude of the  $n^{\text{th}}$  harmonic (consider the “first harmonic” to be at the fundamental frequency  $f_0 = \omega_0 / 2\pi$ , the “second harmonic” to be at twice this frequency, and so on), then we adjust the frequency  $f_0$  of the signal generator to  $f_{\text{filter}} / n$  so that the frequency of the  $n^{\text{th}}$  harmonic will be equal to  $f_{\text{filter}}$ . This means that, for example, if the center frequency of the filter were 60 kHz, as mentioned above, then to measure the amplitude of the third harmonic we would tune the signal generator to a frequency of 20 kHz so that the signal’s third harmonic would now be at the filter’s frequency.

#### HOW THE FILTER’S CHARACTERISTICS LIMIT ACCURACY

The Q of the filter determines how many harmonics we can accurately measure. The signal at the filter’s output will be measured using the DAQ with an “AC RMS” voltmeter application, which means that the measured voltage is the square root of the sum of the squares of the voltages of every harmonic component’s output from the filter. If we want an accurate measure of a single harmonic’s amplitude, then the filter’s gain at the frequencies of the nearby harmonics must be much lower than its gain at the desired harmonic’s frequency.

Assume we want to measure the amplitude of a high harmonic of a square wave to an accuracy of 1%. Since the harmonic amplitudes of a square wave fall as  $1/n$  for the odd harmonics (and vanish for the even harmonics), then for large  $n$  the odd harmonics are of nearly equal amplitude and are spaced apart by a frequency of  $2f_0$ , twice the fundamental frequency. Figure 1 on the next page shows the frequency response of our filter for two different Q values: 140 and 280.

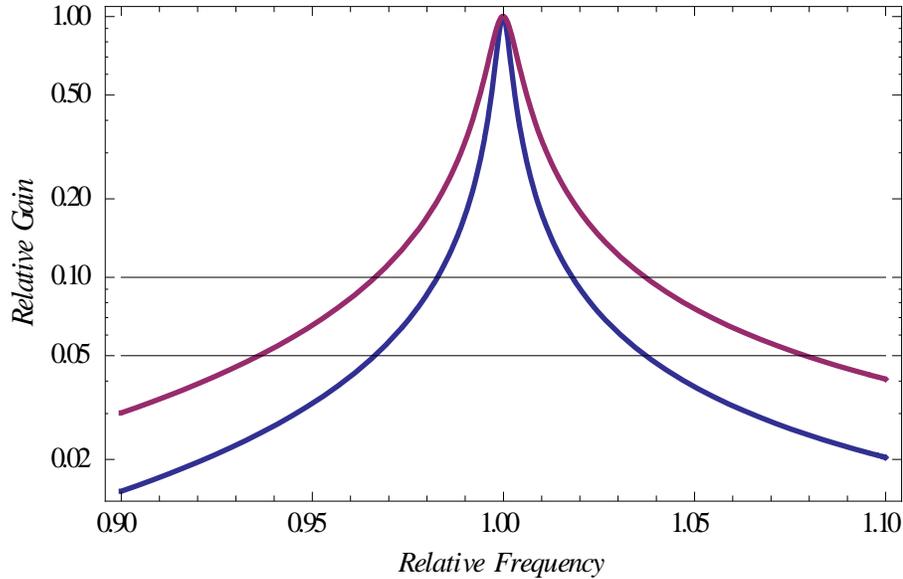


Figure 1  
Relative gains of filters with  $Q = 140$  (upper) and  $Q = 280$  (lower curve)

The gain of the filter falls to 0.1 at frequencies 2% away from the center frequency for the  $Q = 280$  filter and at frequencies 4% away for the  $Q = 140$  filter. Since the voltage we measure is the square root of the sum of the squares of the voltages of the various harmonics, a 0.1 relative gain at a nearby harmonic would result in a 1/2% increase in the measured voltage; one such undesired harmonic on either side of the center frequency would result in a 1% total error. So the nearby harmonics must be no closer to the center frequency of the filter than 2% for the higher- $Q$  example. Since the odd harmonics of a square wave are separated by twice the signal's fundamental frequency, that frequency would have to be no lower than about 1% of the filter frequency; our filter in this case could be centered on the 99<sup>th</sup> harmonic of the signal and still achieve a 1% accuracy! The lower- $Q$  filter shown in Figure 1 would achieve this accuracy for the 49<sup>th</sup> harmonic or lower, where the  $Q = 280$  filter's accuracy would be on the order of 0.1%.

#### AMPLITUDE AND PHASE OF THE HARMONIC MEASUREMENT

In equation (1) we decompose the  $n^{\text{th}}$  harmonic of the signal into a sum of sine and cosine components with amplitudes  $b_n$  and  $a_n$ , respectively. We could choose to write this term as the amplitude and phase of a single sinusoid as follows:

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = A_n \sin(n\omega_0 t + \phi_n), \quad n > 0 \quad (4)$$

Where:

$$A_n^2 = a_n^2 + b_n^2 \quad ; \quad \tan(\phi_n) = a_n / b_n \quad (5)$$

During the experiment a data point will include the measurement of  $V_{\text{RMS}}$ , the RMS voltage at a single harmonic frequency, which corresponds to a measurement of  $A_n$ . Since the harmonic is a sinusoid, this amplitude is related to the measured voltage by:  $A_n = \sqrt{2} \times V_{\text{RMS}}$ . The phase  $\phi_n$  is determined by measuring the phase of the filter output using an oscilloscope or the *Frequency Response* DAQ application. The filter response is such that it shifts the phase forward by  $\pi/2$  at resonance (see Figure 4), so a sine component appears at the filter output as a cosine. As a result,  $\phi_n = \phi_{\text{measured}} - \pi/2$ .

## EXPERIMENT SETUP

The experiment requires a signal generator, filter, DAQ, oscilloscope, and buffer amplifier arranged as in Figure 2. A circuit diagram of the high-Q filter is in Figure 3. You might not have the optional digital voltmeter also shown in the figure.

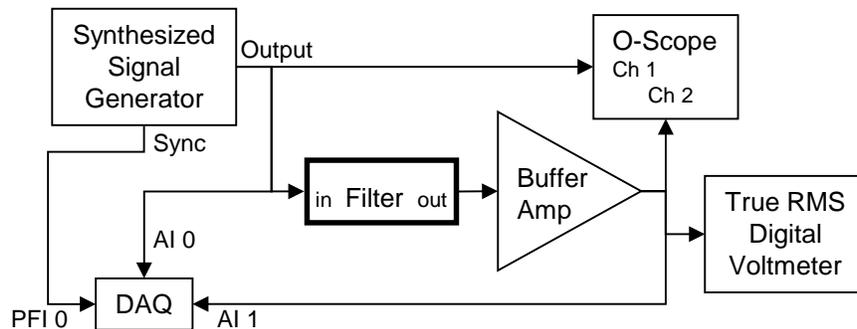


Figure 2  
Equipment arrangement

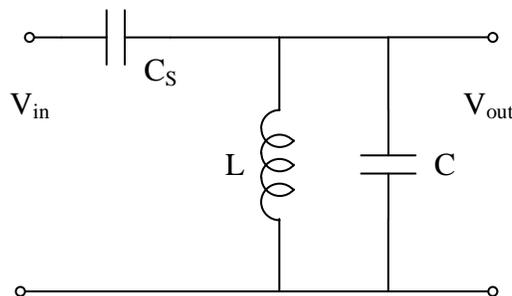


Figure 3  
Filter circuit

The signal generator forms its output waveform using *direct digital synthesis*: the output voltage at any time is generated by a digital-to-analog converter. An internal clock ticks at a high rate, and the generator uses this clock to calculate the instantaneous phase of the desired waveform; the instrument looks up the value of the waveform voltage for this phase and sets the output to this voltage. Because the clock is derived from a very stable, accurate crystal oscillator, the output of the generator is also very stable.

The buffer amplifier has a high input impedance and serves to isolate the filter circuit from any loading caused by the DAQ and the oscilloscope. This is necessary to maintain the filter's  $Q$ . The LC filter itself uses a clever design which results in a  $Q$  higher than you observed in Experiment 2. Referring to Figure 3, the inductor  $L$  is approximately 5mH; the main capacitor  $C$  is approximately 1nF. The source coupling capacitor  $C_S$  is 20pF; the input capacitance of the buffer amplifier and its connecting cable adds another 100pF or so in parallel with capacitor  $C$ . It turns out that all of these capacitances add to form the effective capacitance which determines the resonant frequency of the LC filter.

The gain and phase responses of the filter are shown in Figure 4, where we've assumed a resonant frequency of 60kHz and a  $Q$  of 280. The gain at resonance is approximately 5; note that the gain approaches a constant value of approximately 1/50 for frequencies well above resonance. Appendix A of this experiment analyzes the filter's circuit; make sure you review that section and understand the principles of its design.

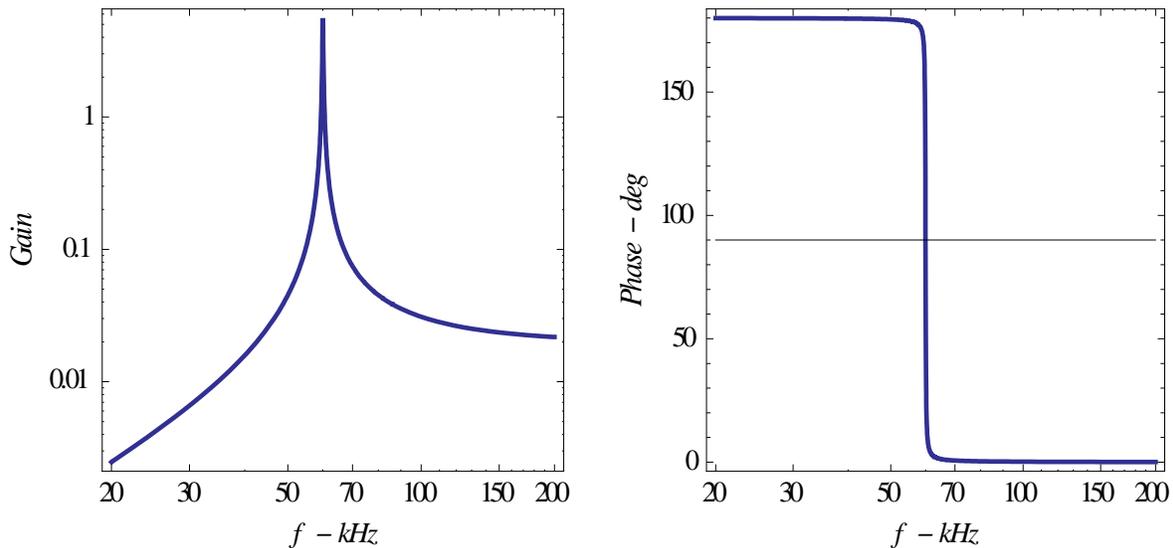


Figure 4  
Gain and phase response of the filter

## PRELAB PROBLEMS

- Express the following waveforms (functions) of Figures 5a and 5b as Fourier series (equation (1)), using the expressions (2) – (3) to calculate the various Fourier coefficients. Each waveform has peak amplitude  $V$  and period  $T$ .

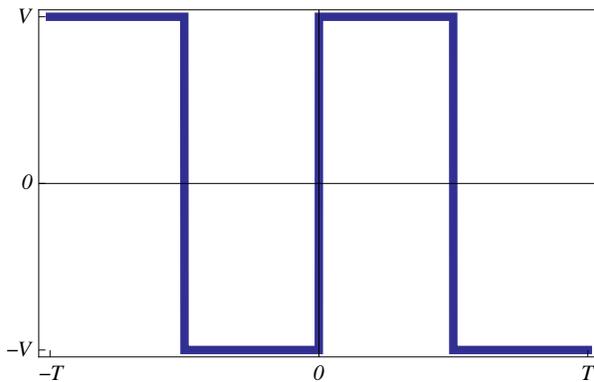


Figure 5a  
Square wave

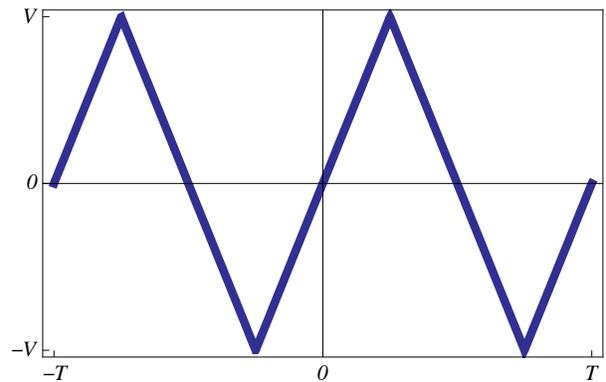


Figure 5b  
Triangle Wave

What is the ratio of the amplitude of the fundamental sinusoid ( $A_1$  in equation (4)) to the peak amplitude of the waveform for the two waves? What is the ratio of the amplitudes of the higher non-zero harmonics to  $A_1$  for each case?

- Show that the RMS voltage of a sinusoid is equal to its amplitude divided by  $\sqrt{2}$ . The RMS value of a periodic function  $f(t)$  is defined to be (where  $T$  is the period):

$$f_{\text{RMS}} = \left[ \frac{1}{T} \int_T f^2(t) dt \right]^{1/2} \quad (6)$$

What are the RMS voltages of the waveforms in Figures 5a and 5b?

- To what frequency would you set the signal generator to measure the amplitude of the 5<sup>th</sup> harmonic ( $A_5$ ) of a waveform if the resonant frequency of the filter is 60 kHz? If the gain of the filter at resonance is 5.40, and the voltmeter reading for this harmonic is 0.382 V, then what is the value for  $A_5$ ? Remember that the voltmeter displays the RMS voltage.

## EXPERIMENT PROCEDURE

1. Assume  $L = 5\text{mH}$ ,  $C = 1.1\text{nF}$  and  $C_s = 20\text{pF}$ . If there is an additional  $100\text{pF}$  of capacitance associated with the buffer amplifier and its input cable, then what should be the approximate resonant frequency of the filter (in Hz)? Starting with this frequency estimate, adjust the signal generator frequency until you find the filter resonance by looking for the maximum output signal amplitude on the oscilloscope.
2. Use the *Frequency Response* application to determine the filter's transfer function around resonance (the signal generator output amplitude should be set to  $100\text{mV}$  pk-pk). Note that the phase at resonance is  $+90$  degrees. Is the resonant frequency compatible with the frequency calculated from the component values? Estimate the Q of the filter by measuring the total frequency width around resonance over which the gain drops by a factor of  $1/\sqrt{2}$ . The Q should be equal to the ratio of the resonant frequency to this frequency width.
3. Record the frequency, input and output peak voltages (amplitudes), and the gain at resonance. Change the signal generator output amplitude by a few  $100\text{ mV}$  and repeat this measurement (don't change the frequency, however). Record the set of measurements for a few different input voltages ranging up to about  $1\text{ V}$  peak. At the higher voltages you may find that the gain changes slightly (and so does the phase). What might be the reason the slight change as the signal voltage is increased?

The ***Transfer Function*** of the filter describes the gain and phase shift introduced by it. You will be able to correct your measured voltage and phase at the filter output to infer what the input voltage and phase are at the filter's resonant frequency by using your knowledge of the Transfer Function. In this way you can calculate the amplitude and phase of each of the harmonics present in an input waveform as you tune them through the filter's resonant frequency.

4. Now pause or quit the *Frequency Response* application and start the *Quad Voltmeter* application. Activate two of the 4 available voltmeters and configure them to take differential measurements of the filter input and output AC RMS voltages. With the signal generator output amplitude set to  $100\text{mV}$  pk-pk and the frequency set to the resonance, compare the ratio of the AC RMS output and input voltages to calculate the filter gain. How does this compare to the gain you measured using the *Frequency Response* application? Note that you can average repeated measurements, which will then also calculate uncertainties in the averages.
5. Set the signal generator to output a square wave. The setup should now give the input square wave RMS voltage and the filter output RMS voltage, which is at the fundamental frequency of the square wave. Use the filter gain to calculate the fundamental ( $1^{\text{st}}$  harmonic) amplitude in the input square wave, and compare to the Fourier series result you calculated in the prelab problem set. Don't forget to include the RMS to amplitude corrections!

6. Measure and record the amplitudes for several odd harmonics of the square wave. Check a few of the even harmonics as well. Check all odd harmonics up to the 15<sup>th</sup>, and then look at a few higher ones up to the 51<sup>st</sup> or so. For the highest harmonics, you may need to increase the signal generator amplitude to keep the output signal from getting too small and noisy. Just remember to record all your RMS voltages (input and filter output).
7. Check two or three of the even harmonic frequencies and examine the waveform at the filter's output using the oscilloscope (you may want to set the scope to average the traces). What is going on? Does the output contain only a single frequency (a sinusoid wave)? Refer again to Figure 4 and think about the many very high harmonics (above the resonant filter frequency) which make up the square wave.
8. Now repeat these measurements using a triangle wave output from the signal generator.

## ANALYSIS

Fit your harmonic amplitude data vs.  $1/n$  and  $1/n^2$  for the square and triangle waves, respectively, and compare to the theoretical values from the Fourier series you calculated.

## APPENDIX A: AN EFFECTIVE HIGH-Q FILTER

*The Tank Circuit Filter*

Figure A1 is a simple circuit which embodies the essential idea of the filter used in Experiment 16: a *tank circuit* driven by an oscillating current source. The “tank” is simply a parallel RLC circuit whose impedance becomes very large at its resonant frequency. Consequently, the voltage across it becomes large as well. Let’s see how this works (you might want to review the appendix to Experiment 2 regarding the *Impedance Concept* before continuing).

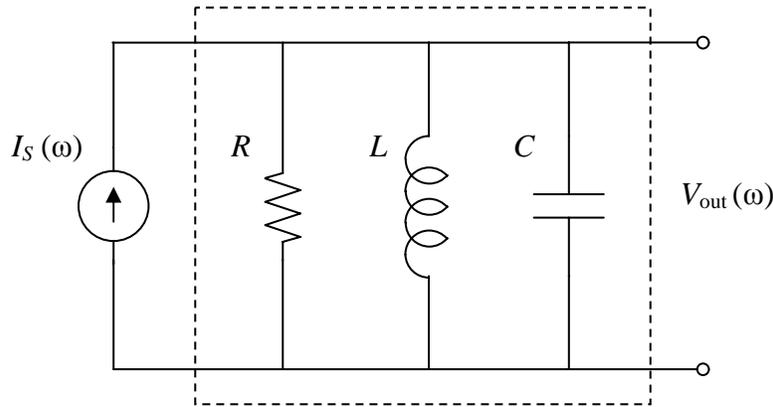


Figure A1

A tank circuit driven by a current source

Since the components are in parallel, their admittances add to give the total admittance of the RLC; the impedance is then the reciprocal of the total admittance:

$$Y_{RLC}(\omega) = Y_R + Y_L + Y_C = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \quad ; \quad Z_{RLC}(\omega) = \frac{1}{Y_{RLC}(\omega)} \quad (\text{A1})$$

At the resonant frequency  $\omega_0 = 1/\sqrt{LC}$ , the impedance has its maximum value and is equal to  $R$ . The  $Q$  of the resonator is the ratio of  $R$  and the characteristic impedance  $Z_0 = \sqrt{L/C}$ , analogous to the resonant circuit of Experiment 2, except for this case high- $Q$  implies  $R \rightarrow \infty$ . In our filter we won’t intentionally lower the  $Q$  by adding in an actual resistor;  $R$  represents the losses in the inductor which limit the  $Q$  we can achieve:

$$R = QZ_0 = Q\sqrt{L/C} \quad (\text{A2})$$

The filter's transfer function (or gain) is thus:

$$\frac{V_{out}(\omega)}{I_S(\omega)} = Z_{RLC} = \left[ \frac{1}{R} + j \left( \omega C - \frac{1}{\omega L} \right) \right]^{-1} \quad (\text{A3})$$

### Using a Voltage Source to Drive the Filter

Implementing the filter of Figure A1 is problematic because it is driven by a *current source* which we must somehow provide. Let us remind ourselves of what we mean by *ideal* and *real* current and voltage sources. Consider a current source: an *ideal current source* would provide the same current to a circuit no matter how high the circuit's impedance might be; the source's output voltage will be as large or as small as necessary to drive the specified current through the load attached to its output. Any *real* current source, of course, has some limit to the voltage it can generate to drive its output current, and the output current will actually vary slightly as the load circuit's impedance rises. We model this behavior by giving the source a finite *source impedance* as shown in Figure A2; similar considerations result in a modification to the ideal voltage source as well.

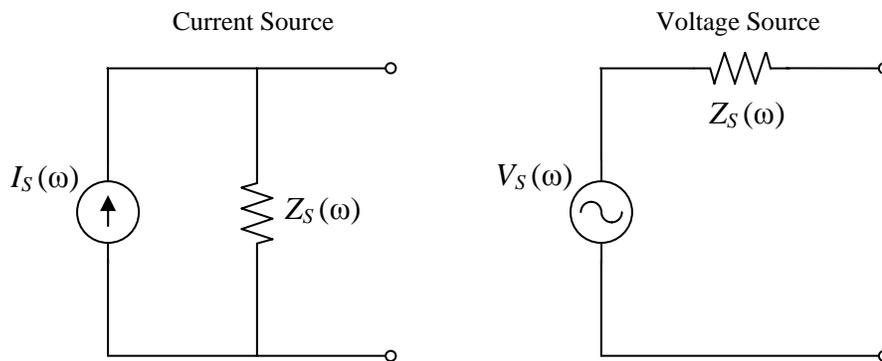


Figure A2  
Models of sources including a finite output impedance

Although the source impedances shown in Figure A2 are drawn as resistors, they could be any combination of circuit elements resulting in a frequency-dependent, complex impedance. The current source model in the figure approaches ideal behavior as  $Z_S(\omega) \rightarrow \infty$ ; the voltage source must have  $Z_S(\omega) \rightarrow 0$  to become ideal. The two models have identical output behaviors if the source impedances are equal and  $V_S(\omega) = Z_S(\omega) I_S(\omega)$ .

Now consider what happens when the ideal current source of Figure A1 is replaced by a real current source with a finite impedance  $Z_S$ : the source's impedance is in parallel with the

components of the RLC filter. If the source impedance is a resistor, then it will lower the  $Q$  of the filter because the parallel combination of it and the filter's  $R = QZ_0$  will lead to a lower total resistance, so that the resulting quality factor,  $Q'$ , will be:

$$\frac{1}{Q'} = \frac{1}{Q} + \frac{1}{Q_s} \quad ; \quad Q_s \equiv \frac{R_s}{Z_0} \quad (\text{A4})$$

This result is an example of a quite general principle regarding the effects on the  $Q$  from a number of sources of loss in the system: the resulting  $Q$  is derived by combining the individual  $Q$  limiters as though they were parallel resistances.

Thus to minimize the effect on the filter's  $Q$ , the source driving the filter must have a very high output resistance, if its impedance is resistive. The signal generator has an output impedance of  $50\Omega$  (resistive), which would completely ruin the filter's response if connected directly to it, since  $Z_0 \approx 2.2\text{ k}\Omega$ . If, on the other hand, the source impedance were a capacitor, then its effect would be to provide a capacitance in parallel with the filter, which would simply increase the total capacitance and change the resonant frequency. Consider the case where we add a small capacitor in series with the output of the signal generator; the resulting source circuit and its equivalent current source are shown in Figure A3.

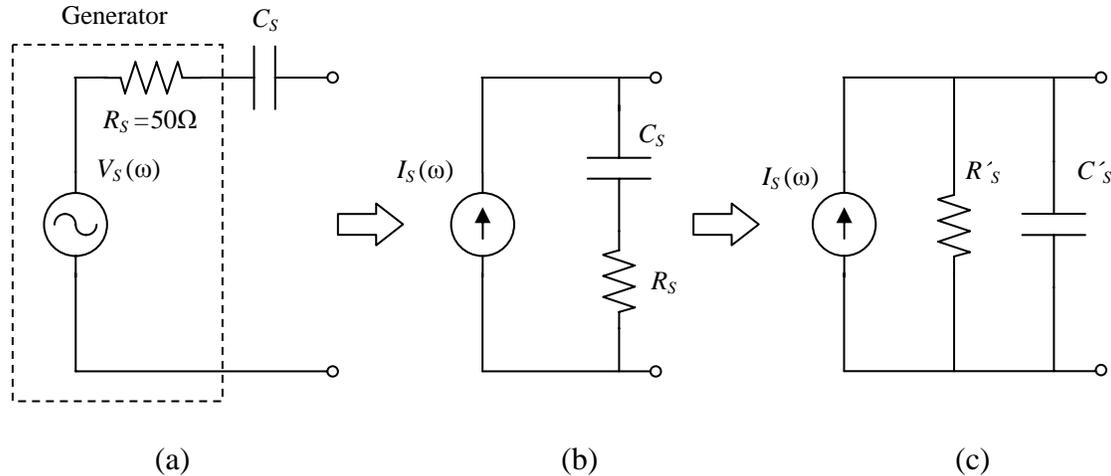


Figure A3

The signal generator with a series capacitor added to its output, and the transformation to an equivalent current source with parallel components

The transformation from (a) to (b) in Figure A3 is straightforward; as remarked previously the impedances of equivalent voltage and current sources are identical. The current source  $I_S(\omega)$  is given by:

$$I_s(\omega) = V_s / (R_s + 1/j\omega C_s) \approx j\omega C_s V_s \quad , \quad \omega \ll 1/R_s C_s \quad (\text{A5})$$

For our circuit  $R_s C_s = (50\Omega)(20\text{pF}) = 10^{-9}$  sec, so at the frequencies we'll be using (60 kHz) the approximation in equation (A5) is very accurate.

For the final transformation from (b) to (c) in Figure A3, we want the parallel combination of  $R'_s$  and  $C'_s$  to give the same total impedance as the series combination of  $R_s$  and  $C_s$ . We equate the impedances and identify the real and imaginary parts as follows:

$$\begin{aligned} R_s + 1/j\omega C_s &= (1/R'_s + j\omega C'_s)^{-1} \quad \rightarrow \quad (R_s + 1/j\omega C_s)^{-1} = 1/R'_s + j\omega C'_s \\ (R_s + 1/j\omega C_s)^{-1} &= \frac{R_s - 1/j\omega C_s}{R_s^2 + (1/\omega C_s)^2} \approx R_s (\omega C_s)^2 + j\omega C_s \\ \therefore C'_s &= C_s \quad ; \quad R'_s = (1/\omega C_s)^2 / R_s = |Z_{C_s}|^2 / R_s \end{aligned} \quad (\text{A6})$$

Where we've again made use of  $\omega \ll 1/R_s C_s$  in deriving (A6). So the capacitor in the parallel equivalent (c) in Figure A3 is the same as the series capacitor in (a), and the parallel resistor  $R'_s$  is very large in value if the series resistance  $R_s$  is small compared to the impedance of  $C_s$ . This is certainly the case for  $R_s = 50\Omega$  and  $C_s = 20\text{pF}$  at a frequency of 60 kHz: with these values equation (A6) results in  $R'_s > 3 \times 10^8 \Omega$ . Neglecting the small effect of  $C_s$  on the  $Z_0$  of the LC resonator,  $Q_s = R'_s / Z_0 \approx (3 \times 10^8) / (2 \times 10^3) \approx 10^5$ , which has a completely negligible effect on the Q of the filter (see equation (A4)).

### *Frequency Response of the Capacitively-Coupled Filter*

By using a small capacitor to couple the output of the signal generator to the filter's tank circuit we avoid loading the filter and maintain its high-Q performance. This technique is an example of *weak-coupling* to a system in order to avoid contamination of its inherent behavior by the other elements of the experiment's apparatus. This principle finds wide application in the study of many physical systems — quantum-mechanical systems in particular.

Now synthesize the results of (A3), (A5), and (A6) to write down the gain of the capacitively-coupled filter circuit with our signal generator providing a voltage to the filter of  $V_{in}(\omega)$ , the filter with characteristics Q and  $Z_0$ , using components  $L$  and  $C$ , and with coupling capacitor  $C_s$ :

$$\frac{V_{out}}{V_{in}} = \frac{j\omega C_s}{\frac{1}{QZ_0} + j \left[ \omega(C + C_s) - \frac{1}{\omega L} \right]} \quad (\text{A7})$$

At resonance the imaginary part of the denominator of (A7) vanishes. The plots in Figure 4 illustrate the response function of equation (A7). Here we summarize the major characteristics of the response:

$$\begin{aligned} \text{Gain}(\omega_0) &= jQ \frac{C_s}{C + C_s} \quad ; \quad \omega_0 = \frac{1}{\sqrt{L(C + C_s)}} \\ \text{Gain}(\omega \gg \omega_0) &= \frac{C_s}{C + C_s} \quad ; \quad \text{Gain}(\omega \ll \omega_0) = -\omega^2 LC_s \end{aligned} \quad (\text{A8})$$

We isolate the filter from any significant additional loss (and reduction of Q) at  $V_{out}$  by using a buffer amplifier with an input resistance of more than  $10^{10}\Omega$ . The amplifier and the cable connecting it to the filter introduce some parallel capacitance, however, so the capacitance  $C$  must include this additional term. Finally, the inductor has inter-winding capacitance of several picoFarads which also adds to  $C$ .