Appendix D: The Wave Vector

In this appendix we briefly address the concept of the *wave vector* and its relationship to traveling waves in 2- and 3-dimensional space, but first let us start with a review of 1-dimensional traveling waves.

1-dimensional traveling wave review

A real-valued, scalar, uniform, harmonic wave with angular frequency ω and wavelength λ traveling through a lossless, 1-dimensional medium may be represented by the real part of the complex function $\psi(x,t)$:[1]

$$\psi(x,t) = \psi(0,0)\ \exp(ikx - i\omega t) \tag{D-1}$$

where the complex phasor $\psi(0,0)$ determines the wave's overall amplitude as well as its phase ϕ_0 at the origin $(x,t)=(0,0)$. The harmonic function's *wave number k* is determined by the wavelength λ:

$$k = \pm 2\pi/\lambda \tag{D-2}$$

The sign of k determines the wave's direction of propagation: $k > 0$ for a wave traveling to the right (increasing x). The wave's instantaneous phase $\phi(x,t)$ at any position and time is

$$\phi(x,t) = \phi_0 + (kx - \omega t) \tag{D-3}$$

The wave number k is thus the spatial analog of angular frequency ω: with units of radians/distance, it equals the rate of change of the wave's phase with position (with time t held constant), i.e.

$$k = \frac{\partial \phi}{\partial x}; \quad \omega = -\frac{\partial \phi}{\partial t} \tag{D-4}$$

(note the minus sign in the differential expression for ω). If a point, originally at $(x = x_0, t = 0)$, moves with the wave at velocity $v_\phi = \omega/k$, then the wave's phase at that point will remain constant:

$$\phi(x_0 + v_\phi t,\ t) = \phi_0 + k(x_0 + v_\phi t) - \omega t = \phi_0 + kx_0$$

The velocity v_ϕ is called the wave's *phase velocity*.

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[1] In this appendix we use the physicists' sign convention for the exponential argument: $i(kx - \omega t)$, rather than the electrical engineering convention $j(\omega t - kx)$ used in General Appendix A and Experiment 14.

The function $\psi(x,t)$ is a solution of the 1-dimensional *wave equation*

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{v_\phi^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{D-5}$$

and the function $\Psi(x) = e^{i\omega t}\psi(x,t) = \psi(0,0)\exp(ikx)$ is a solution of the 1-dimensional *Helmholtz equation*

$$\frac{d^2\Psi}{dx^2} + k^2\Psi = 0 \tag{D-6}$$

The functions $\psi(x,t)$ and $\Psi(x)$ represent *harmonic solutions* to equations (D-5) and (D-6) because they have well-defined frequency ω and wave number k. Of course, there are an infinite number of other valid solutions to equations (D-5) and (D-6): the appropriate solution will depend on a particular problem's boundary value constraints and might be constructed from a linear superposition of harmonic solutions with various values for k (or ω) using Fourier analysis.

Waves in 2 and 3 dimensions

To extend the 1-dimensional harmonic wave $\psi(x,t)$ given by (D-1) to a wave in multi-dimensional space, consider the expression (D-3) for the wave's phase ϕ. Using Cartesian coordinates so that the position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, we could simply add a linear phase term for each coordinate:

$$\begin{gathered}\phi(\vec{r},t) = \phi_0 + (k_x x + k_y y + k_z z - \omega t) = \phi_0 + (\vec{k}\cdot\vec{r} - \omega t) \\ \vec{k} \equiv k_x\hat{x} + k_y\hat{y} + k_z\hat{z}\end{gathered} \tag{D-7}$$

Thus the equivalent expression in two or more spatial dimensions for the uniform, harmonic wave (D-1) becomes

$$\psi(\vec{r},t) = \psi(\vec{0},0)\ \exp(i\vec{k}\cdot\vec{r} - i\omega t) \tag{D-8}$$

where, again, $\psi(\vec{0},0)$ is a complex phasor which determines the amplitude and phase offset of $\psi(\vec{r},t)$. The vector $\vec{k}$ is called the *wave vector*, the multi-dimensional analog of the wave number k in (D-1).

From the phase expression (D-7), $\vec{k}$ is evidently the position gradient of the phase, which turns out to be a good, general definition of the wave vector:

$$\boxed{\vec{k} = \nabla\phi(\vec{r},t)} \tag{D-9}$$

Equation (D-9) is obviously simply the generalization of the expression for k in (D-4) to multi-dimensional spaces. If the wave vector $\vec{k}$ is constant and uniform throughout 3-dimensional space, then $\psi(\vec{r},t)$ becomes a *plane wave*: so named because its instantaneous loci of uniform phase, which by equation (D-9) must be everywhere perpendicular to $\vec{k}$, form a set of parallel planes; in 2-space the analogous loci comprise the set of lines everywhere perpendicular to $\vec{k}$ (see Figure D-1).

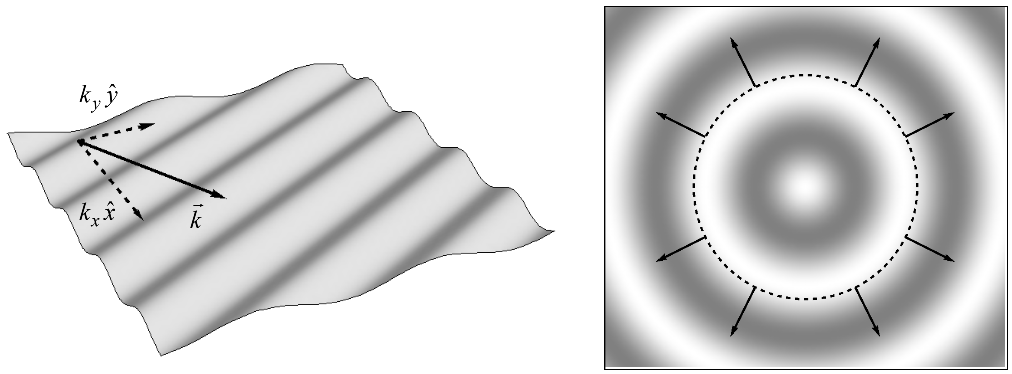

Figure D-1: (Left) a transverse plane wave propagating along a surface in the direction given by its wave vector $\vec{k}$. Note that the magnitudes of the wave vector's components give the rate of change of the wave's phase along their respective directions. (Right) a circular wave propagating outward. A selection of wave vectors is shown; each vector is perpendicular to the surface of constant phase at its location (in this case, a circle centered on the source).

Because the wave vector is the position gradient of the phase of a wave, its dot product with a unit vector gives the rate of change of the phase along that unit vector's direction (in radians/length). This is illustrated in the left-hand graphic in Figure D-1 for the two unit vectors $\hat{x}$ and $\hat{y}$. Generally, the wave vector varies from place to place, as shown in the right-hand graphic in Figure D-1; we then refer to the *vector field* $\vec{k}(\vec{r})$. Only if the wave vector $\vec{k}$ is uniform throughout space do we get a plane wave, equation (D-8).

The phase velocity of a plane wave (equation (D-8), with uniform $\vec{k}$) is parallel to $\vec{k}$ and is again given by $v_\phi = \omega/k$

$$\vec{v}_\phi = \frac{\omega}{k}\hat{k} \tag{D-10}$$

The multi-dimensional wave equation satisfied by $\psi(\vec{r},t)$ is (we assume the medium is *isotropic*)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{v_\phi^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

or (D-11)

$$\nabla^2\psi - \frac{1}{v_\phi^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

The associated Helmholtz equation is, naturally,

$$\nabla^2\Psi + k^2\Psi = 0 \tag{D-12}$$

As for the 1-D case, general solutions of either of these differential equations may be constructed from Fourier sums or integrals of plane-wave solutions with various values of $\vec{k}$. Such constructions are appropriate if the boundary conditions are planar (3-D case) or defined along straight lines (2-D case). Such boundary value problems over finite volumes will restrict the choices of wave vectors $\vec{k}$ which will satisfy them. The acceptable values of $\vec{k}$ become a set of *eigenvalues* for a particular boundary value problem, and the associated functions $\psi_{\vec{k}}(\vec{r},t)$ may be used to construct the system's *normal modes*.

Relation to standing waves in a cavity

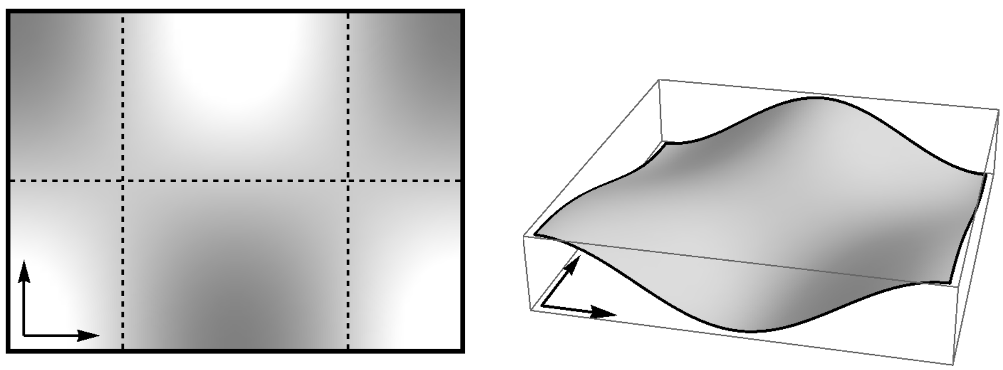

Figure D-2: Two views of a standing wave mode in a rectangular resonant cavity with "free" (Neumann) boundary conditions. The nodal lines of the mode are shown (dashed) in the left graphic; the arrows in each graphic represent $\hat{x}$ and $\hat{y}$ unit vectors. In this example such solutions may be constructed by adding a symmetric set of plane waves all with the same wave number $|\vec{k}|$ as described in the text and shown in Figure D-3.

The boundary conditions for a *resonant cavity* will usually demand that the Helmholtz equation's wave solutions form *standing waves* rather than a single travelling wave with a definite direction of propagation; Figure D-2 provides an example of a standing wave solution in a rectangular cavity such as that used in Experiment 15.

In the case of the rectangular cavity shown in Figure D-2, standing waves with the proper symmetries and boundary values may be synthesized by adding pairs of oppositely-directed travelling waves, all with a common wave number $k = |\vec{k}|$, as shown in Figure D-3 at right. For the solution shown in Figure D-2, each plane wave must have $|k_x| = 2\pi/l_x$ and $|k_y| = \pi/l_y$, where l_x and l_y are the lengths of the horizontal and vertical sides of the cavity, respectively. Additionally, all four plane waves must have phase $\phi = 0$ at the lower-left corner of the cavity for the resulting standing wave to satisfy the cavity's boundary

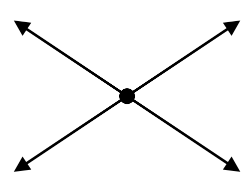

Figure D-3: Wave vectors of the four traveling waves whose sum provides the standing wave solution shown in Figure D-2.

conditions. The Experiment 15 notes cover this situation more thoroughly.

The symmetries of a particular boundary value problem involving the wave equation or its related Helmholtz equation may point to a different choice for the set of elementary solutions $\psi(\vec{r},t)$ of the differential equation, rather than using plane waves. For example, a 2-dimensional system with circular symmetry is usually solved in the most straightforward manner by choosing circularly-symmetric solutions whose radial variation is given by an appropriate member of the family of *Bessel* and related functions. For any such function representing a travelling wave with well-defined frequency ω (a harmonic function), however, the wave vector $\vec{k}$ at any position $\vec{r}$ is still given by the gradient of the function's phase with $\vec{r}$ (equation (D-9) and the right-hand graphic in Figure D-1).

Circularly-symmetric cavities resonate with standing waves displaying radial + circular nodal patterns as shown in Figure D-4. Unlike the rectangular cavity solutions, however, these patterns cannot be generated from a finite sum of plane waves. The most straightforward analytic (closed-form) expression for the standing wave solution is as the product of a function for the radial variation (a Bessel function) and the azimuthal variation (a sinusoid). For the standing wave shown in Figure D-4, the solution of equation (D-12) turns out to be:

$$\Psi(r,\theta) = J_2(kr)\cos(4\pi\theta) \quad \text{(D-13)}$$

where J_2 is the 2$^{\text{nd}}$-order *Bessel function of the first kind*, and the wave number k is such that $J_2(ka)$ is the *second zero* of J_2 with a = the circular cavity's radius, or $k \approx 8.41724/a$.

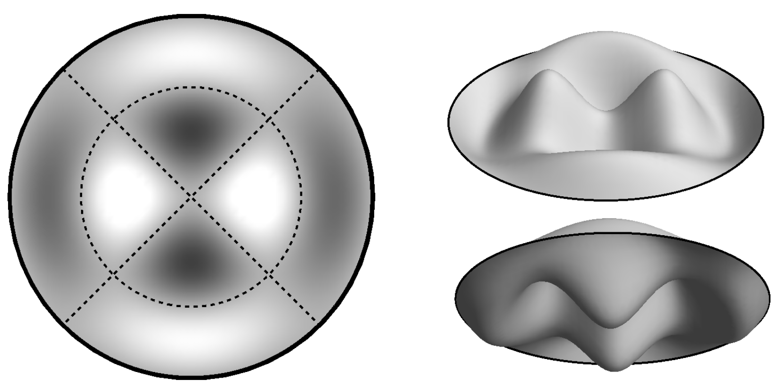

Figure D-4: Views of a particular standing wave mode in a circular, 2-D membrane with a "fixed" (Dirichlet) boundary condition, such as a drumhead. The nodal lines of the mode are shown (dashed) in the left graphic. The right-hand images show very exaggerated surface displacements of the membrane, seen from above and below.