General Appendix A
Transmission Line Resonance due to Reflections (1-D Cavity Resonances)

1. *Waves Propagating on a Transmission Line*

General

A transmission line is a 1-dimensional medium which can support the propagation of waves in either direction along its length. Transmission lines may be used to model the behaviors of many physical systems, including organ pipes, optical fibers, and thin, stretched, elastic strings in addition to the typical electrical examples of coaxial cables and microwave waveguides. A ***linear*** medium allows the construction of complicated wave solutions by superposing several simple waves such as sinusoids at various frequencies — a linear transmission line supports the simultaneous propagation of many waves with various frequencies and in either direction along it. The instantaneous displacement at time t for any point on the line is simply the sum of the displacements due to all the various waves at that point and at that time. This discussion will assume that the transmission line is linear.

We shall use the phasor notation described in Experiment 2, Appendix A, to express the solutions of our transmission line problem (a frequency representation). A sinusoid with angular frequency ω and wave number k propagating on the line in the $+z$ direction varies with position and time as

$$\begin{gathered} a(z,\, t) = \mathrm{Re}\left[A(0)\, e^{-\alpha z}\, e^{j(\omega t - k z)}\right] \\ \text{or}:\; a(z,\, t) = \mathrm{Re}\left[A(z)\, e^{j\omega t}\right],\ \text{where}:\; A(z) = A(0)\, e^{-(jk+\alpha) z} \end{gathered} \tag{1}$$

As in Experiment 2, we use the engineering phase convention (to convert to the physics phase convention, simply replace the imaginary constant j with $-i$). The phasor $A(z)$ describes the variation in phase and amplitude of the wave with distance along its direction of propagation. The ***phase velocity*** of the wave is given by $v_\phi = \omega/k$ and is determined by the material characteristics of the transmission line and the physical nature of the wave propagating along it. Generally, this velocity is a function of the frequency ω (or, equivalently, the wave number k). Transmission lines with v_ϕ independent of frequency are called ***nondispersive***.

The exponential term $e^{-\alpha z}$ in (1) represents the attenuation of the wave due to losses in the transmission line. The (real, nonnegative) loss parameter α has units of $(\text{length})^{-1}$; the distance $\lambda = 1/(2\,\alpha)$ is sometimes called the ***attenuation length*** of the transmission line, because the intensity of the wave (which goes as $A(z)^2$) decreases by a factor of e^{-1} for each additional distance of λ along the line. If the line is lossless (or ***ideal***), then $\alpha = 0$. In general, α varies with frequency (or k), usually increasing at higher frequencies. It is often more convenient to define an alternative, dimensionless loss parameter $\delta = \alpha/k$ so that

$$A(z) = A(0)\, e^{-jk(1-j\delta) z} \tag{2}$$

with both $k(\omega)$ and $\delta(\omega)$ being real, nonnegative functions of frequency. It may turn out that δ is a more slowly varying function of k (or ω) than is α. In the discussion that follows we assume that δ is generally small and is only weakly dependent on k. Of course, for an ideal transmission line $\delta = 0$. The loss parameter δ is related to the so-called ***loss tangent*** of a transmission medium, and, as we shall see later on, the ***quality factor*** Q of a cavity resonator constructed from the transmission line is often given by $Q = 1/(2\,\delta)$.

Propagator Notation

The exponential term in equation (2) tells how the phasor of a wave at some position on a transmission line is related to its phasor at a different position. The distance parameter z increases in the same direction as the wave propagates. It is convenient to define the ***propagation operator,*** or ***propagator,*** $\mathcal{P}(z)$ as this exponential term:

$$\mathcal{P}(z) \equiv e^{-j\,k\,(1-j\,\delta)\,z} \tag{3}$$

$$A(x+d) = \mathcal{P}(\pm\,|\,d\,|)\,A(x) \tag{4}$$

Choose the positive sign in the propagator's argument if you are propagating the phasor in the direction of the wave propagation and the minus sign for transforming a phasor backward (opposite to the wave's direction). Some properties of the propagator (all easily derivable from equation (3)):

$$\begin{gathered} \mathcal{P}(0) = 1 \\ \mathcal{P}(d_1)\,\mathcal{P}(d_2) = \mathcal{P}(d_1 + d_2) \\ \mathcal{P}(-d_1) = 1/\mathcal{P}(d_1) \end{gathered} \tag{5}$$

One important note about (4) and (5): it is assumed that k and δ are fixed at known values for all such expressions. If there are waves with different frequencies being included in a single expression, then one must use a notation such as $\mathcal{P}(d;\omega)$ or write out the exponential term explicitly.

Characteristic Impedance

For an electrical transmission line comprised of two conductors (such as a coaxial cable or the lumped-parameter line of Experiment 14), we shall consider the amplitude phasor $A(z)$ to represent the voltage difference between the two conductors at position z due to the propagation of a wave, so that $A(z) = V(z)$. The current flows in the two conductors are equal and opposite, and the direction of the current flow in the more positive conductor is the same as the direction of the wave's propagation. $V(z)$ is proportional to the current phasor $I(z)$; this proportion defines the transmission line's ***characteristic impedance*** Z_0 (6). The geometry and materials of the line determine the value of Z_0, just as they do the phase velocity v_ϕ and the loss parameter δ.

$$\begin{gathered} \text{For a single wave with frequency } \omega: \\ V(z)/I(z) \equiv Z_0(\omega) \\ P(z) = \frac{1}{2}\,\mathrm{Re}\,[V(z)\,I\,(z)^*] = \frac{1}{2}\,|V(z)|^2\,\frac{\mathrm{Re}\,[Z_0(\omega)]}{|Z_0\,(\omega)|^2} \end{gathered} \tag{6}$$

A wave transmits power along the line in its direction of propagation. This power flow is given by $P(z)$ in the second of equations (6). The characteristic impedance Z_0 is generally a function of frequency, as indicated. For the line to represent a realizable physical system, Z_0 must have a nonnegative real part. If $Z_0(\omega)$ is imaginary, then $V(z)$ and $I(z)$ are 90° out of phase, and a wave cannot propagate along the line at that frequency (the frequency is beyond the ***cutoff frequency*** for the line). For the discussion that follows, we assume that Z_0 is real and positive, which is a very good approximation for nearly ideal transmission lines ($\delta \ll 1$) in the frequency range where waves can propagate. Note that a wave has an *energy density* (energy per unit length) which is proportional to its squared amplitude; the power $P(z)$ is the flow of this energy across the point z. The wave's energy density propagates along the transmission line at the *group velocity*: $v_G = d\,\omega/d\,k$. For a *dispersive* transmisssion medium it is generally true that $v_G \neq v_\phi$.

Now consider the case of two waves with the same frequency ω propagating in opposite directions on a transmission line at a position x, the right-going wave with voltage phasor $a(x)$ and the left-going wave with voltage phasor $b(x)$ at that position. The resulting voltage phasor at x is given by the sum of these two wave voltages, $V(x) = a(x) + b(x)$. The currents due to the two waves flow in opposite directions, so that the total current at x is given by the difference in the two waves' currents. Each wave current is related to its wave voltage by the relation in (6), so the voltage and current at a point on the line are given by

$$\begin{gathered}\text{For right-going wave } a(x) \text{ and left-going wave } b(x):\\ V(x) = a(x) + b(x) \\ I(x) = a(x)/Z_0 - b(x)/Z_0 = (a(x) - b(x))/Z_0\end{gathered} \tag{7}$$

Note that in this case (two oppositely propagating waves), the voltage and current phasors at a point are not simply related by Z_0, so that $V(x) \neq Z_0\, I(x)$.

Reflection at a Termination

The wave $a(x)$ propagates to the right on a line with real, nonnegative characteristic impedance Z_0 and encounters a termination of the line at position x_0. At this point the line is terminated by an impedance Z, as shown in figure 1. The voltage across Z and the current through it are $V(x_0)$ and $I(x_0)$, respectively. Ohm's law requires that $V(x_0) = Z\, I(x_0)$, but the voltage and current due to the wave are $a(x_0)$ and $a(x_0)/Z_0$. If $Z \neq Z_0$, then the boundary condition on the voltage and current at x_0 requires that a wave $b(x)$ propagate away from the termination; then (7) may be applied to satisfy the relationship between $V(x_0)$ and $I(x_0)$ imposed by the impedance Z.

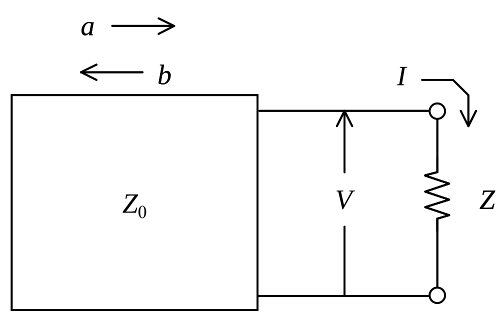


Figure 1. Wave reflection at the termination of a transmission line.

If we assume that $b(x_0) = \Gamma\, a(x_0)$, then for $V(x_0) = Z\, I(x_0)$ to be satisfied, we find (using (7)) that the ***reflection coefficient*** Γ must be given by the expression in equation (8). Also listed with this equation are some commonly encountered values for Z and the corresponding values of Γ.

$$\Gamma = \frac{Z - Z_0}{Z + Z_0} \tag{8}$$

$$\begin{gathered}\text{Short: } Z = 0 \rightarrow \Gamma = -1 \\ \text{Open: } Z = \infty \rightarrow \Gamma = 1 \\ \text{Terminated: } Z = Z_0 \rightarrow \Gamma = 0\end{gathered}$$

Next consider a termination at the point $x = 0$ which contains a voltage source V_S, used to inject a wave $c(x)$ onto a semi-infinite transmission line (figure 2a). The source impedance is Z and the line characteristic impedance is Z_0, as shown. First assume that there is no wave approaching the termination from the right, so that the only wave on the line is $c(x)$, emitted from the termination (because of its embedded voltage source). As far as the source is concerned, the transmission line looks like a resistor equal to Z_0. A voltage divider is formed by this resistance and the source impedance Z, so the voltage across the transmission line is

$$c(0) = \frac{Z_0}{Z + Z_0}\, V_S = \frac{1 - \Gamma}{2}\, V_S \tag{9}$$

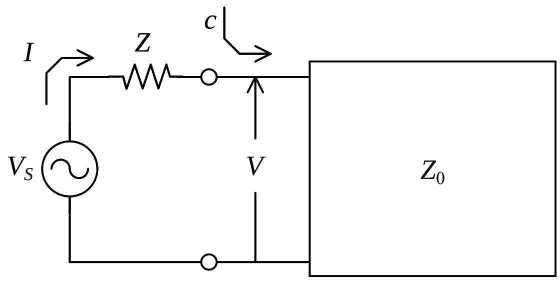
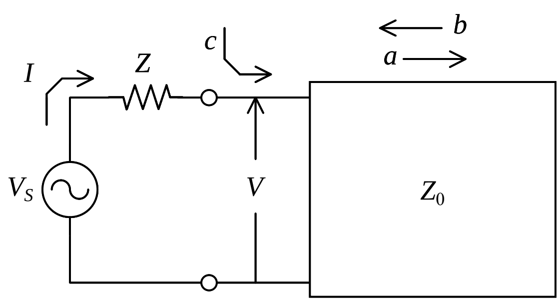


Figure 2a. A termination which includes a source.

where we've also used equation (8) to express Z in terms of the reflection coefficient Γ. In figure 2a, $c(0) = V$ because $c(x)$ is the only wave present on the line. Figure 2b includes an incoming wave $b(x)$. The outgoing wave $a(x)$ includes the reflection of $b(0)$ from the termination impedance Z in addition to the source contribution c. The solution is easily found (because the circuit is linear) and is given in (10).



$$\begin{gathered} c = \frac{1 - \Gamma}{2}\, V_S\,; \quad a(0) = c + \Gamma\, b(0) \\ V = a(0) + b(0) = c + (\Gamma + 1)\, b(0)\,; \quad I = \frac{a(0) - b(0)}{Z_0} = \frac{c + (\Gamma - 1)\, b(0)}{Z_0} \end{gathered} \tag{10}$$

Figure 2b. A termination which includes a source and an incoming wave.

2. *1-D Cavity Resonator*

General Solution

A finite segment of transmission line terminated at both ends forms a 1-dimensional cavity for waves propagating on the line. Any losses in the cavity, either due to a nonzero loss parameter δ or a reflection coefficient Γ at either end with $|\Gamma| < 1$ would imply that the power in a wave in the cavity would eventually dissipate. If the termination at one end of the line includes a source, then a wave c may be injected at that end, driving waves in the cavity at the frequency of the source. This arrangement is shown in figure 3 for a line of unit length driven by a source in the left-hand termination (where we take the position parameter $x = 0$).

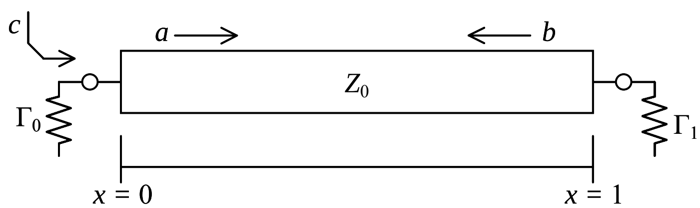


Figure 3. A unit-length, driven, 1-D cavity.

To solve this system, we use the propagator and terminator equations to connect the wave amplitudes $a(x)$ and $b(x)$:

$$\begin{gathered} a(0) = c + \Gamma_0\, b(0)\,;\; a(1) = \mathcal{P}(1)\, a(0) \\ b(1) = \Gamma_1\, a(1)\,;\; b(0) = \mathcal{P}(1)\, b(1) \\ a(x) = \mathcal{P}(x)\, a(0)\,;\; b(x) = \mathcal{P}(1-x)\, b(1) = \mathcal{P}(-x)\, b(0) \end{gathered} \tag{11}$$

$$\begin{gathered} \therefore a(0) = c + \Gamma_0\, \mathcal{P}(1)\, \Gamma_1\, \mathcal{P}(1)\, a(0) = c + \Gamma_0\, \Gamma_1\, \mathcal{P}(2)\, a(0) \\ a(0) = c\, (1 - \Gamma_0\, \Gamma_1\, \mathcal{P}\,(2))^{-1} \end{gathered}$$

$$\frac{a(x)}{c} = \frac{\mathcal{P}(x)}{1 - \Gamma_0\, \Gamma_1\, \mathcal{P}(2)}\;;\; \frac{b(x)}{c} = \frac{\Gamma_1\, \mathcal{P}(2-x)}{1 - \Gamma_0\, \Gamma_1\, \mathcal{P}(2)} \tag{12}$$

The total amplitude of the cavity oscillation (voltage for the electrical transmission line), $V(x) = a(x) + b(x)$:

$$\begin{gathered} \frac{V(x)}{c} = \frac{a(x) + b(x)}{c} = \frac{\mathcal{P}(x) + \Gamma_1\, \mathcal{P}(2-x)}{1 - \Gamma_0\, \Gamma_1\, \mathcal{P}(2)} = \frac{\mathcal{P}(x-1) + \Gamma_1\, \mathcal{P}(1-x)}{\mathcal{P}(-1) - \Gamma_0\, \Gamma_1\, \mathcal{P}(1)} \\ \frac{V\,(x)}{V(0)} = \frac{\mathcal{P}\,(x) + \Gamma_1\, \mathcal{P}\,(2-x)}{1 + \Gamma_1\, \mathcal{P}(2)} = \frac{\mathcal{P}(x-1) + \Gamma_1\, \mathcal{P}(1-x)}{\mathcal{P}(-1) + \Gamma_1\, \mathcal{P}(1)} \end{gathered} \tag{13}$$

The final expressions in (13) were obtained by multiplying through by $\mathcal{P}(-1)/\mathcal{P}(-1)$; their symmetry will make the expressions in the next section easy to derive. Note also that $V(x)/V(0)$ does not depend on Γ_0, because using $V(0)$ as a reference effectively establishes a perfect short ($\Gamma_0 = -1$) at the source end, with $c = V(0)$. You can confirm this by comparing the expression for $V(x)/V(0)$ to that for $V(x)/c$ with $\Gamma_0 = -1$.

Shorted or Open Termination at the Right End - Resonance

Consider the case where the termination opposite the driven end is a perfect short ($\Gamma_1 = -1$) or a perfect open ($\Gamma_1 = +1$). The propagator $\mathcal{P}(z)$ is an exponential function (equation (3)), so that

$$\begin{aligned} \mathcal{P}(-z) + \mathcal{P}(z) &= 2\cos\left[k\,(1 - j\,\delta)\,z\right] \\ \mathcal{P}(-z) - \mathcal{P}(z) &= 2\,j\sin\left[k\,(1 - j\,\delta)\,z\right] \end{aligned} \tag{14}$$

Substituting (14) into the expression (13) for $V(x)/V(0)$ with $\Gamma_1 = \pm 1$ gives

$$\Gamma_1 = -1 \text{ (short)}: \quad \frac{V(x)}{V(0)} = \frac{\sin\left[k\,(1 - j\,\delta)\,(1 - x)\right]}{\sin\left[k\,(1 - j\,\delta)\right]} \tag{15}$$

$$\Gamma_1 = +1 \text{ (open)}: \quad \frac{V(x)}{V(0)} = \frac{\cos\left[k\,(1 - j\,\delta)\,(1 - x)\right]}{\cos\left[k\,(1 - j\,\delta)\right]} \tag{16}$$

If δ is small, then for $k = n\,\pi/2$ ($n = 1, 2, \ldots$) the denominator of either (15) or (16) becomes small ($\sim j\,k\,\delta$, equation (16) for n odd, (15) for even n); while the numerators will be ~ 1 depending on the position x. For the other parity of n (odd in (15), even in (16)), however, the denominator is ~ 1. This means that both the shorted and open lines will exhibit multiple resonances, each line having adjacent resonances spaced by $\Delta\,k \approx \pi$. Figure 4 shows an example of the first few resonances for the open termination line (equation (16)).

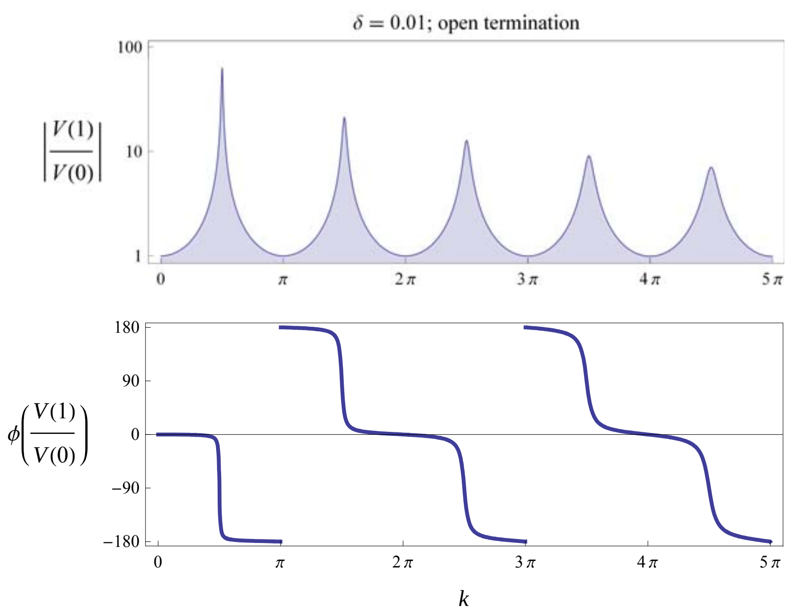


Figure 4. Typical gain and phase response of a unit-length, 1-D cavity, showing the first few evenly-spaced resonances (log scale).

The quality factor Q of a resonator is given by $Q = \text{(energy stored)}/\text{(energy lost/radian of phase in time)}$, or, since $\omega = d\phi/dt$ and $v_\phi = \omega/k$,

$$Q = \frac{E}{dE/d\phi} = \frac{\omega E}{dE/dt} = \frac{v_\phi\, k\, E}{dE/dt} \tag{17}$$

$$E_a + E_b = E = \int dE = \int_0^1 \frac{\partial E}{\partial x}\, dx = \int_0^1 \frac{P(x)}{v_G}\, dx = \frac{1}{v_G} \int_0^1 [P_a(x) + P_b(x)]\, dx \tag{18}$$

where the integral in (18) is over the volume of the cavity (figure 3), and the power $P_a(x) + P_b(x)$ is the sum of the powers in the two oppositely-directed traveling waves a and b in the cavity. Note also that the energy density in a wave is related to its power by the group velocity and not the phase velocity: $\partial E/\partial x = P(x)/v_G$. The energy dissipated in the cavity is being supplied by the power source at $x = 0$, so in equation (17) $dE/dt = \mathrm{Re}\,[V(0)\, I(0)^*]/2 = P_a(0) - P_b(0)$. The wave power is given by the formula in (6), where the wave's voltage $V(x)$ is its amplitude, so this power is, of course, proportional to the square of the magnitude of the wave amplitude, e.g. $P_a(x) \propto |\, a(x)\,|^2$. The wave b is just the lossless reflection of wave a at the right end of the cavity, after which it passes through the cavity again. Clearly, as far as the attenuation of the waves due to losses is concerned, this situation is equivalent to a single wave traveling through an identical transmission line twice as long as the cavity, with an amplitude attenuation as it travels of $e^{-\alpha x} = e^{-k\,\delta\, x}$; the squared amplitude therefore decays as $e^{-2\,\alpha\, x} = e^{-2\,k\,\delta\, x} = P_a(x)/P_a(0)$. This also implies that $P_b(0) = P_a(2)$, since we assume that $|\,\Gamma_1\,| = 1$. Equation (17) becomes

$$Q = \frac{v_\phi\, k\, E}{dE/dt} = \frac{v_\phi}{v_G} \frac{k}{P_a(0) - P_b(0)} \int_0^2 P_a(x)\, dx = \frac{v_\phi}{v_G} \frac{k}{1 - P_a(2)/P_a(0)} \int_0^2 \frac{P_a\,(x)}{P_a(0)}\, dx$$

$$\therefore\ Q = \frac{v_\phi}{v_G} \frac{k}{1 - e^{-4\,k\,\delta}} \int_0^2 e^{-2\,k\,\delta\, x}\, dx = \frac{1}{2\,\delta} \frac{v_\phi}{v_G} \tag{19}$$

If the transmission medium is nondispersive, then $v_G = v_\phi$ and $Q = 1/(2\,\delta)$, as mentioned previously.

Approximate Solutions for Small δ

Consider the shorted- and open-end solutions (15) and (16). The functions can be expanded using the following identities:

$$\sin(A - j\,B) = \cosh B \sin A - j \sinh B \cos A \approx \sin A - j\,B \cos A,\ \ B \ll 1$$
$$\cos(A - j\,B) = \cosh B \cos A + j \sinh B \sin A \approx \cos A + j\,B \sin A,\ \ B \ll 1$$

The approximations in the above equations are to first order in B. If δ is small, then we can substitute these approximations in the numerators and denominators of equations (15) and (16), which gives the following approximate expressions, with numerators and denominators each accurate for $k\,\delta \ll 1$:

$$\Gamma_1 = -1\ \text{(short)}:\ \frac{V\,(x)}{V(0)} \approx \frac{k\,\delta\ (1 - x) \cos[k\,(1 - x)] + j\ \sin[k\,(1 - x)]}{k\,\delta \cos k + j\ \sin k},\ \ k\,\delta \ll 1 \tag{20}$$

$$\Gamma_1 = +1\ \text{(open)}:\ \frac{V\,(x)}{V(0)} \approx \frac{\cos[k\,(1 - x)] + j\,k\,\delta\,(1 - x) \sin[k\,(1 - x)]}{\cos k + j\,k\,\delta \sin k},\ \ k\,\delta \ll 1 \tag{21}$$

These expressions can be simplified considerably at their respective resonant frequencies — $k_n = n\pi$ for (20), $k_n = (n - 1/2)\pi$ for (21), n a positive integer — resulting in a single expression for $V(x)$, as long as the appropriate resonant value for k_n is used, as shown in equation (22) and figure 5. The relative error in (22) is of order $(k_n \delta)^2$.

$$\text{At resonance}: \frac{V(x)}{V(0)} \approx (1-x)\cos(k_n x) - j\frac{1}{k_n \delta}\sin(k_n x),\ \ k_n \delta \ll 1 \tag{22}$$

$$k_n = \begin{cases} n\pi, & \text{if } \Gamma_1 = -1 \text{ (short)} \\ (n - 1/2)\pi, & \text{if } \Gamma_1 = +1 \text{ (open)} \end{cases},\ \ n \text{ a positive integer}$$

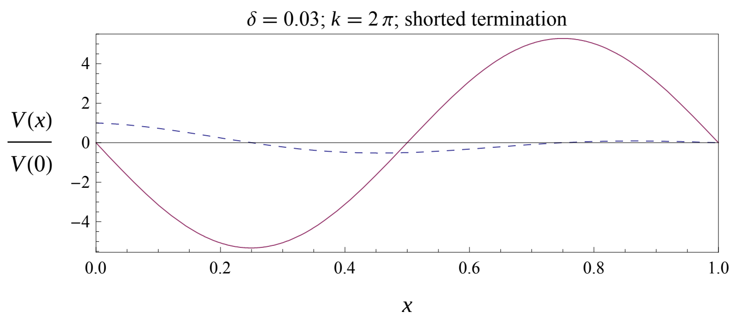


Figure 5. Typical complex response vs. position of a unit-length, 1-D cavity at its second resonance. Imaginary (solid) and real (dashed) parts are plotted.

So at resonance the ***quadrature-phase*** part of the response phasor $V(x)$ varies as the sine of x, with amplitude $V(0)/(k_n \delta)$. The ***in-phase*** amplitude decreases from $V(0)$ to 0 at the far end of the line ($x = 1$). If $k_n \delta \ll 1$, then the quadrature phase component dominates the phasor amplitude, so the overall amplitude varies as $\sin x$, to an accuracy of $\sim (k_n \delta)^2$, except near the nodes (0's of the quadrature phase part), where the phasor amplitude is $(1 - x)V(0)$.

Consider now the response at $x = 1$ of the open-termination resonator, equation (21), which we choose because the open end of the transmission line is always an anti-node of the response at resonance.

$$\frac{V(1)}{V(0)} \approx \frac{1}{\cos k + j k \delta \sin k},\ \ k\delta \ll 1$$

$$\frac{V(1)}{V(0)} \approx \frac{1}{(1-k\delta)\cos k + k\delta(\cos k + j\sin k)} = \frac{1}{(1-k\delta)\cos k + k\delta\, e^{jk}} = e^{-jk}\left(\frac{1}{k\delta + e^{-jk}(1-k\delta)\cos k}\right) \tag{23}$$

The leading $\exp(-jk)$ term in (23) is just the phase delay due to the wave propagation down the line from $x = 0$ to $x = 1$. The second term in the expression should capture the rapid changes in amplitude and phase of the response near resonance; at resonance $\cos k = 0$, and the magnitude of the response is $1/(k\delta)$, as in equation (22) for $x = 1$. To proceed further we wish to simplify the model of the resonant response in (23) as a function of k for values near a resonant frequency $k_n = (n - 1/2)\pi$. We define $\Delta k = k - k_n$, $\Delta k \ll 1$ and expand the expression about k_n to get rid of the transcendental functions, discarding terms of order $(\Delta k)^2$ and $(\Delta k)\delta$ or higher:

$$e^{-jk}(1-k\delta)\cos k \approx e^{-j(k_n+\Delta k)}(1-k_n\delta)\cos(k_n+\Delta k) \approx j\Delta k,\ \ \Delta k \ll 1 \text{ and } k_n = (n - 1/2)\pi$$

$$\therefore \frac{V(1)}{V(0)} \approx j\,(-1)^n \left(\frac{1}{k_n\,\delta + j\,\Delta k} \right) \; ; \; k\,\delta \ll 1,\; \Delta k = k - k_n \ll 1 \;\text{(open-termination resonator)} \tag{24}$$

where in (24) the overall phase term $\exp(-j\,k)$ in (23) has been replaced by $\exp(-j\,k_n) = j\,(-1)^n$. The squared magnitude of the response (24) is called a ***Lorentzian response*** and is typical of many resonant systems, including the natural line shapes of emission lines due to electron transitions in excited atoms. Equations (25) and (26) provide the magnitude and phase of the Lorentzian:

$$\text{with: } k_n = (n - 1/2)\,\pi,\; \text{open termination},\; \Delta k = k - k_n$$

$$\left| \frac{V(1)}{V(0)} \right|^2 \approx \frac{1}{(\Delta k)^2 + (k_n\,\delta)^2} \tag{25}$$

$$\arg\left(\frac{V(1)}{V(0)} \right) \approx -\left(k_n + \tan^{-1}\left(\frac{\Delta k}{k_n\,\delta} \right) \right) \equiv \frac{\pi}{2}\,(-1)^n - \tan^{-1}\left(\frac{\Delta k}{k_n\,\delta} \right) \tag{26}$$

When $\Delta k / k_n = \pm\delta$, the response magnitude (25) is half its maximum value of $1 \big/ (k_n\,\delta)^2$, and the arctangent in (26) has a value of $\pm(\pi/4)$. These are consistent with a resonant response with $Q = 1/(2\,\delta)$, as was found previously. Incidentally, the '$\equiv$' symbol in (26) means congruence modulo 2π, wrapping the phase into the branch $-\pi < \phi \le \pi$, as in figure 4.

The Lorentzian response described by equations (25) and (26) are easily extended to apply to $V(x)/V(0)$ generally for a resonance of either the shorted or open lines by using (22):

$$\text{with: } k_n = \begin{cases} n\,\pi, & \text{if } \Gamma_1 = -1 \text{ (short)} \\ (n - 1/2)\,\pi, & \text{if } \Gamma_1 = +1 \text{ (open)} \end{cases},\; n \text{ a positive integer},\; \Delta k = k - k_n$$

$$\left| \frac{V(x)}{V(0)} \right|^2 \approx \frac{\sin^2(k_n\,x)}{(\Delta k)^2 + (k_n\,\delta)^2} \tag{27}$$

$$\arg\left(\frac{V(x)}{V(0)} \right) \approx -\left(\arg\left(j\,\sin(k_n\,x)\right) + \tan^{-1}\left(\frac{\Delta k}{k_n\,\delta} \right) \right) \tag{28}$$

3. *Lossless Cavities with Partial Reflections at the Boundaries*

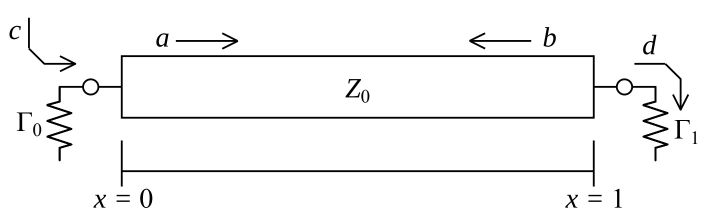


Figure 6. A unit-length, driven, 1-D cavity with lossy terminations.

Now consider the case where the transmission medium is lossless, but the terminations have $|\Gamma| < 1$. The situation is depicted in figure 6. For example, a Fabry-Perot optical cavity may have identical end mirrors which reflect nearly all of the incident light, with but a small fraction transmitted through. Another example would be a section of lossless transmission line whose characteristic impedance Z_0 differs substantially from that of the media interfaced to either end. In particular, we will be interested in not only how the wave amplitude within the cavity varies with frequency, but also the frequency variation of the power transmitted through the cavity (injected at one end and absorbed at the other).

A wave c is injected at the left end of the cavity; the wave at the right end is partially reflected, but a portion d is absorbed by the terminator at that end. The two end termination reflection coefficients $|\Gamma_0| \lesssim 1$ and $|\Gamma_1| \lesssim 1$, i.e. just a little less than 1. The propagator equations representing the relationships among the various quantities are still given by (12), repeated below as equation (29).

$$\frac{a(x)}{c} = \frac{\mathcal{P}(x)}{1 - \Gamma_0\,\Gamma_1\,\mathcal{P}(2)}\;;\;\frac{b(x)}{c} = \frac{\Gamma_1\,\mathcal{P}(2-x)}{1 - \Gamma_0\,\Gamma_1\,\mathcal{P}(2)} \tag{29}$$

For a lossless medium the propagator is particularly simple: $\mathcal{P}(z) = e^{-j\,k\,z}$; thus its only effect is to shift the wave's phase, and $|\,\mathcal{P}(z)\,| = 1$ for all z.

For the sake of argument, assume that the end reflection coefficients are nearly perfect shorts, so that each $\Gamma = -1 + \epsilon$ and for some small complex number ϵ: $|\,\epsilon\,| \ll 1$. Using the equations (29) the power absorbed by the right-hand termination in figure 6 is proportional to $|\,d\,|^2 = |\,a(1)\,|^2 - |\,b(1)\,|^2 = \left(1 - |\,\Gamma_1\,|^2\right)|\,a(1)\,|^2$. Using our approximation for Γ:

$$\frac{|\,d\,|^2}{|\,a(1)\,|^2} = 1 - (-1 + \epsilon^*)\,(-1 + \epsilon) = 2\,\mathrm{Re}\,(\epsilon) - |\,\epsilon\,|^2 \approx 2\,\mathrm{Re}\,(\epsilon) \tag{30}$$

Of course, we must have $\mathrm{Re}\,(\epsilon) > 0$ for this case so that $|\,\Gamma_1\,| < 1$. Resonances occur for those values of the wave number k which make the denominators in (29) small. It is straightforward to show that the minimum magnitude of the denominator is $2\,|\,\epsilon\,|$, occuring for $k_n = \pi\,n$ (n a whole number). Since for our lossless medium $|\,a(0)\,|^2 = |\,a(1)\,|^2$, the escaping wave d is related to the source wave c at resonance by:

$$\frac{|\,d\,|^2}{|\,c\,|^2} = \frac{|\,d\,|^2}{|\,a(1)\,|^2}\,\frac{|\,a(0)\,|^2}{|\,c\,|^2} \approx \frac{\mathrm{Re}\,(\epsilon)}{2\,|\,\epsilon\,|^2} \tag{31}$$

Thus for real, positive ϵ the escaping wave power grows by a factor of $1/(2\,\epsilon)$ near resonance. Assuming that ϵ is not a function of k, then the wave number dependence of $|\,d\,|^2$ is (again, assuming real, positive $\epsilon \ll 1$)

$$\frac{|\,d\,|^2}{|\,c\,|^2} \approx \frac{2\,\epsilon}{|\,1 - \Gamma_0\,\Gamma_1\,e^{-2\,j\,k}\,|^2} \approx \frac{\epsilon}{2\left(\epsilon^2 + \sin^2 k\right)} = \frac{1}{1 + \left(\sin^2 k\right)/\epsilon^2} \times \frac{1}{2\,\epsilon} \tag{32}$$

So at resonance ($k = n\,\pi$), the power gain $|\,d\,|^2 \big/ |\,c\,|^2 = 1/(2\,\epsilon)$, but quickly falls to $\epsilon/2$ as k moves away from resonance ($\sin^2 k \sim 1 \gg \epsilon^2$). The intensity of the wave d falls to half its peak value as k moves away from resonance by an amount that makes $\sin k = \pm\epsilon$. For small ϵ, the change in k is simply $\Delta\,k = \pm\,\epsilon$, so the full width at half-maximum of each resonance peak is 2ϵ. As with the case of the lossy transmission line discussed previously, the response near a resonance is approximately Lorentzian:

$$\frac{|\,d\,|^2}{|\,c\,|^2} \approx \frac{\epsilon/2}{(\Delta\,k)^2 + \epsilon^2} \tag{33}$$

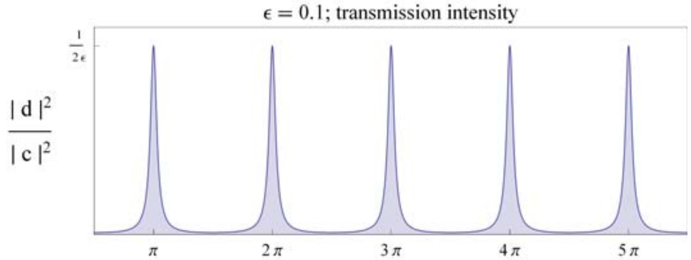


Figure 7. Typical transmission intensity vs. k of a cavity with slightly leaky terminations.