Experiment 5

Resonant circuits and active filters

Now we return to the realm of linear analog circuit design to consider the final op-amp circuit topic of the term: *resonant circuits* and *active filters*. Two-port networks in this category have transfer functions which are described by linear, second-order differential equations.

First we investigate how a bit of positive feedback may be added to our repertory of linear op-amp circuit design techniques. We consider a *negative impedance* circuit which employs positive feedback in conjunction with negative feedback. This sort of circuit is found in a wide variety of linear op-amp applications including amplifiers, *gyrators* (inductance emulators), current sources, and, in particular, resonant circuits and sinusoidal oscillators.

Next we switch topics to consider the archetypal resonant circuit: the *LC* resonator (inductor + capacitor). We use this circuit to define the *resonant frequency* and *quality factor* for a second-order system, and we investigate the frequency and transient responses of a high-*Q, tuned circuit*. We then introduce the general topic of second-order filters: resonant circuits with quality factors of around 1. We describe the behavior of second-order low-pass, high-pass, and band-pass filters.

Finally, we implement such filters using linear op-amp circuits containing only *RC* combinations in their feedback networks, eliminating the need for costly and hard-to-find inductors. The filters' circuitry will employ positive as well as negative feedback to accomplish this feat. We discuss some of the tradeoffs when selecting the *Q* to use in a second-order filter, and look at Bessel and Butterworth designs in particular.

As a postscript, the *More circuit ideas* section presents, among other things, a couple of sinusoidal oscillators constructed from resonant circuits.

CONTENTS

TABLE OF CIRCUITS

POSITIVE FEEDBACK IN LINEAR APPLICATIONS

The hallmark of linear op-amp circuit design is the use of negative feedback to maintain the two op-amp inputs at the same voltage, as described all the way back in Experiment 1. When we used positive feedback in Experiment 4 it was to create a Schmitt trigger, a highly nonlinear circuit in which the op-amp output spends its time in either positive or negative saturation (except during those brief, slew-rate limited transitions from one saturation limit to the other), and the two op-amp inputs are generally at very different voltages. In this section we consider linear analog circuits which, nevertheless, include a bit of positive feedback along with a strong dose of negative feedback. By using this technique we can design analog circuits with a greatly expanded range of capabilities, as we shall soon see.

The Howland current pump

A linear op-amp circuit employing both positive and negative feedback is the *Howland current pump*, invented many years ago by Bradford Howland at MIT. This circuit is an example of *a voltage to current converter*: it establishes a constant current through a load which is proportional to an input control voltage.

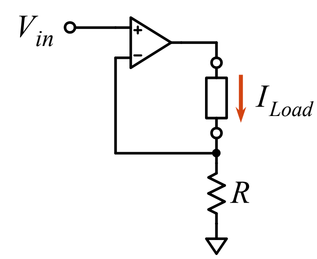
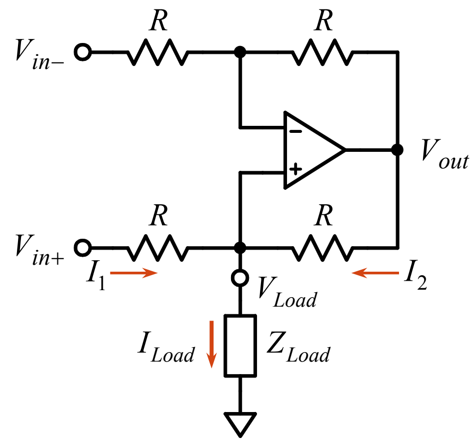


Figure 5-1: A simple voltage to current converter: $I_{Load} = V_{in}/R$.

First, however, recall the simple circuit in Figure 5-1, originally introduced in Experiment 1. In this case the load element is positioned in the op-amp's negative feedback loop. Because the two op-amp input voltages are equal the supplied input voltage V_{in} also appears across the resistor R. The op-amp output current required to establish this voltage across R must also pass through the load, so the load current must be $I_{Load} = V_{in}/R$, and it will be independent of any variation in the load's impedance.

A significant drawback of the simple circuit of Figure 5-1, however, is that its load must *float*: the load's two terminals must both be independent of the circuit ground. A more common requirement which our simple circuit can't satisfy is to supply a fixed current into a single terminal of a load that itself might be part of a complicated sub-circuit; the injected current would then be returned through the system ground or power supplies. The Howland Current Pump shown in Figure 5-2 on page 5-2 provides an elegant solution to this design problem. To understand how the circuit of Figure 5-2 operates, we'll use the principle of linear superposition presented in Experiment 1.

Figure 5-2: The *Howland current pump* supplies a current into a terminal of a grounded load which is independent of the load's impedance: it implements a *current source* which adjusts the load voltage V_{Load} as necessary to maintain the selected load current I_{Load}. The selected load current is determined by the difference in the input voltages V_{in+} and V_{in-} and is given by equation 5.1; its operation is described in the text.



Because the ideal op-amp inputs draw no current, it must be true that the load current is given by $I_{Load} = I_1 + I_2$. To determine I_1 and I_2 we will analyze the circuit in two stages:

(1) connect the V_{Load} terminal directly to ground (so that V_{Load} is 0) and determine I_1 and I_2 as a function of the input voltages V_{in+} and V_{in-};

(2) next demonstrate that the sum $I_{Load} = I_1 + I_2$ is independent of V_{Load}, and therefore is independent of the load impedance Z_{Load}.

First consider step (1). Setting $V_{Load} = 0$ grounds the op-amp *+Input*, so the op-amp part of the circuit becomes a simple inverting amplifier with a gain of −1, and the op-amp's $V_{out} = -V_{in-}$. With the V_{Load} terminal connected to ground, voltages V_{in+} and V_{out} then also appear across the two lower resistors R, determining currents I_1 and I_2. Thus, in this case the load current I_{Load} must be:

5.1
$$I_{Load} = I_1 + I_2 = (V_{in+} - V_{in-})/R$$

Now for step (2): consider the effect of a change in the load voltage on the load current. Because the circuit is linear, it is sufficient to consider the simple case wherein $V_{in+} = V_{in-} = 0$ and then determine how I_{Load} depends on V_{Load}; from the results of step (1) we know that $I_{Load} = 0$ if $V_{Load} = 0$ in this case. What we want to show is that $I_{Load} = 0$ is maintained even when $V_{Load} \neq 0$. In any case V_{Load} is the voltage at the op-amp's *+Input*. Since the terminal at V_{in-} is now ground, the op-amp circuit is a pure noninverting amplifier with gain 2 for any voltage at its *+Input*, so $V_{out} = 2V_{Load}$. Since the terminal at V_{in+} is ground, the voltage drop across the bottom-left resistor in Figure 5-2 is $V_{in+} - V_{Load} = -V_{Load}$, and therefore $I_1 = -V_{Load}/R$. The voltage across the bottom-right resistor is $V_{out} - V_{Load} = V_{Load}$, so $I_2 = V_{Load}/R$. This means that $I_{Load} = I_1 + I_2 = 0$, and the load current remains 0 even when there is a nonzero load voltage. By linear superposition, this means that even a nonzero load current calculated using (5.1) must be unaffected by changes in V_{Load}. Therefore equation (5.1) is independent of the load impedance Z_{Load}. The Howland current pump serves as a *current source* whose output is programmed by its differential input voltage ($V_{in+} - V_{in-}$). Note that our derivation, however, assumes that the resistor values are all well-matched. In particular, this means that the sources of the input voltages V_{in+} and V_{in-} must have low output impedances, so voltage followers will probably be required to buffer these inputs.

More importantly, our analysis of the Howland current pump circuit has implicitly assumed that the op-amp can maintain its two input terminals, +*Input* and −*Input*, at identical voltages as it responds to changes in V_{in+} and V_{in-} as well as to changes in the load impedance Z_{Load}. But the op-amp's output is fed back to its +*Input* via the bottom-right resistor in Figure 5-2, so the circuit employs positive as well as negative feedback. Then how do we know that the op-amp's output voltage won't just run off to saturation, as in the case of a Schmitt trigger circuit? In the next section we investigate this potential difficulty.

Stability of linear circuits which include positive feedback

Consider the generic analog op-amp circuit shown at right. The feedback network is constructed from of a set of four impedances forming voltage dividers from the op-amp's output V_{out} back to both the op-amp's +*Input* and its –*Input*. Call the fractions of V_{out} fed back to these two op-amp inputs f_+ and f_-, respectively; then for this example

$$f_+ = Z_{i+}\big/\left(Z_{i+} + Z_{f+}\right),\quad f_- = Z_{i-}\big/\left(Z_{i-} + Z_{f-}\right).$$

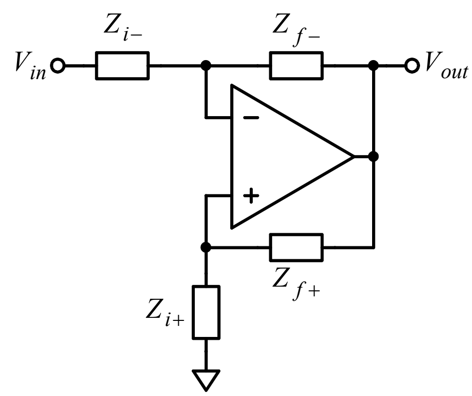


Figure 5-3: Generic inverting amplifier with some positive feedback as well.

At zero frequency (DC), both f_+ and f_- are nonnegative real numbers, so as long as $f_- > f_+$, the net feedback will be negative and the circuit will remain stable and linear (note that we must include the output impedance of the source for V_{in} when calculating f_+ and f_-). At other frequencies, the stability criterion is a little more subtle: one must find the set of solutions $s = j\omega$ of the equation:

5.2 $$f_-(j\omega) \;-\; f_+(j\omega) \;+\; 1\big/g(j\omega) \;=\; 0$$

where $g(j\omega)$ is the open-loop gain function of the op-amp (as described in Experiment 2). We replace all occurrences of $j\omega$ in equation (5.2) with the complex variable s and then find the complex solutions s of (5.2). If the real parts of the various solutions of (5.2) are all negative, i.e. $\mathrm{Re}(s) < 0$ for all roots s of this equation, then the circuit will be stable. Proof of this criterion is left to the exercises (just kidding!). Actually, proof of this theorem belongs to the general field of *control system engineering*.

The criterion expressed above always gives a correct assessment of a feedback circuit's stability, and it is generally the best way to evaluate our systems. A simpler, more naïve criterion is often adequate for our elementary circuits, and it is more intuitive and easier to apply:

> ***Stability criterion when including positive feedback***
>
> As long as the fraction of the op-amp output fed back through the *negative* feedback loop is greater than the fraction fed back through the *positive* feedback loop, then the circuit will probably remain stable and predictable.
>
> *This criterion must hold for all input and load impedances the circuit may encounter and for* ***all frequencies*** *the op-amp is capable of amplifying (up to the op-amp's gain-bandwidth product), not only the set of frequencies you plan to input to the circuit.*

A negative impedance circuit

With this stability criterion in mind, consider the circuit in Figure 5-4, a one-port network driven by a source with output impedance Z_s, as shown. What we are interested in is to determine the circuit's input impedance, $Z_{in} = V_{in}/I_{in}$, and also to decide how the op-amp's two input terminals should be connected in order to ensure the circuit's stability.

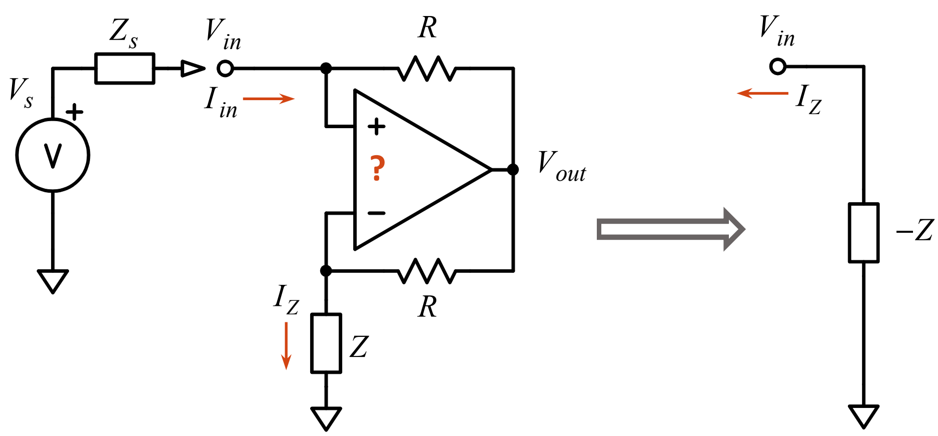


Figure 5-4: A *negative impedance* circuit. As explained in the text, the input impedance of the op-amp circuit, V_{in}/I_{in}, is $-Z$, the *negative* of the impedance of the element Z in the circuit. The "?" on the op-amp symbol implies that *we must carefully consider how the op-amp's inputs should be connected* — with *–Input* at top as shown, or should the op-amp's inputs be flipped? This issue is discussed in the text.

If we assume that the stability criterion is satisfied, i.e. that there is net negative feedback around the op-amp, then we can assume that the ideal op-amp's +*Input* and −*Input* voltages are equal. This means that the input voltage V_{in} appears at both op-amp inputs, and, therefore, that V_{in} must be the voltage drop across the impedance Z (since its other end is at ground). The resulting current through Z (I_Z in the figure) must come from the op-amp output by flowing through the feedback resistor. The voltage drops across the two resistors R in the figure are the same: $V_{out} - V_{in}$, so current I_Z must also flow through the top resistor from the op-amp output toward the input terminal of the circuit. Thus $I_{in} = -I_Z$ (surprise!). Therefore:

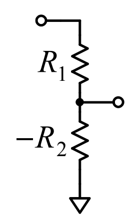
5.3 $$\boxed{Z_{in} = V_{in}/I_{in} = -V_{in}/I_Z = -Z}$$

The circuit's input impedance is the *negative* of the impedance Z — we have a device with a *negative impedance* to ground. Thus, for example, if the impedance Z is a simple resistor, *then when we apply a positive voltage to the circuit's input, the circuit will push current back into the source*; it would be a *source* of power rather than an *absorber* of power (of course, the extra power ultimately comes from the op-amp's power supplies).

Before we consider how to use this circuit for something interesting, we must make sure that our original assumption is correct: is the stability criterion satisfied? We must check that there is less output feedback to the op-amp's *+Input* than to its *–Input*. Since the two feedback resistors each have value R in Figure 5-4, the feedback fraction will depend on the impedance to ground at each op-amp terminal, since that impedance will set the voltage divider ratio at that terminal. For the *–Input* terminal in Figure 5-4 this is the impedance Z; the impedance to ground at the op-amp's *+Input* is the *output impedance of the driving source*, Z_s.

> Thus for the configuration shown in Figure 5-4, the stability criterion requires that $|Z_s| < |Z|$ at all frequencies the op-amp can amplify. If the opposite were true, i.e. $|Z_s| > |Z|$, then the inputs to the op-amp should be reversed.

Now let's consider a simple application of our negative impedance circuit: a voltage divider which includes a "resistor" with a *negative* value, as shown at right. The gain of this circuit is $V_{out}/V_{in} = -R_2/(R_1 - R_2)$, which has a magnitude greater than 1 if $R_1 < 2R_2$. We could build this sort of amplifier using our negative impedance circuit to realize the negative resistance R_2; we just have to be careful as we consider the stability criterion in this case.



The gyrator

An interesting application of the negative impedance circuit is to try to find a more complicated circuit using it which *inverts* an impedance: $Z \to 1/Z$; then we could emulate an inductor using a capacitor ($1/j\omega C \to j\omega C$). Such a device is called a *gyrator*. A circuit which accomplishes this task using voltage dividers and our negative impedance circuit is shown in Figure 5-5 on page 5-6. Working out that this circuit does indeed invert the impedance Z is left as an exercise to the reader. What is more problematic, however, is to decide how to orient the op-amps' inputs and apply the stability criterion to this circuit.

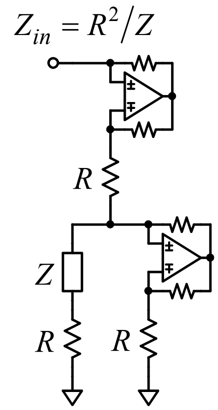


Figure 5-5: A simple *gyrator* built from negative impedance converter circuits. Deciding how to orient the op-amp inputs could be a problem, however.

A more sophisticated application of combined positive and negative feedback to implement a gyrator circuit is shown in Figure 5-6. This circuit is quite different from the negative impedance circuit, and it is quite stable, though we won't consider its stability in detail here.

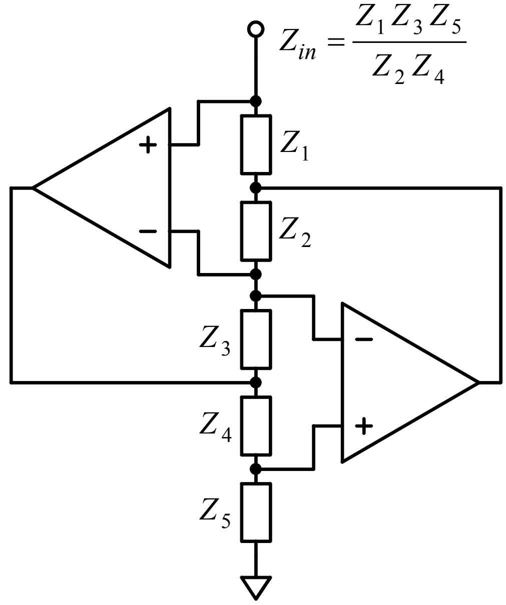


Figure 5-6: A practical gyrator circuit. The op-amps will maintain their inputs at the same potential, so the voltage across Z_5 and the voltage at the junction of Z_2 and Z_3 must both equal the input voltage. With this condition it is straightforward to derive the input impedance formula shown.

Assuming that the op-amps are ideal and working linearly and correctly, then their inputs must all be at the same potential, which must be the input voltage, since an op-amp input is connected there (Figure 5-6). With this constraint plus the observation that the current through Z_2 must equal that through Z_3, and similarly for the currents through Z_4 and Z_5, it is straightforward to show that the circuit's input impedance is given by the expression included in the figure.

> The conventional way to emulate an inductor with the gyrator circuit in Figure 5-6 is to use a capacitor for Z_2 and resistors for the other impedance elements.

SECOND-ORDER SYSTEMS AND RESONANCE

LC resonant amplifier

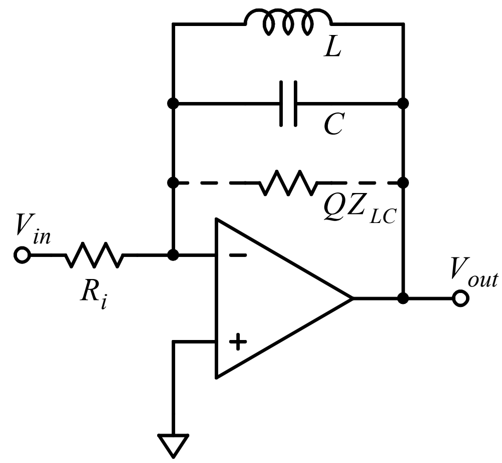


Figure 5-7: An inverting amplifier with a parallel *LC* combination forming the feedback impedance. In addition the circuit will include an equivalent parallel resistance along with the *L* and *C*. Its equivalent value QZ_{LC} is addressed in the text.

Consider the inverting amplifier circuit in Figure 5-7 and assume for the moment that the op-amp is ideal. Note that the feedback impedance is given by the parallel combination of an inductor L and a capacitor C; in addition there is a resistance to which we assign the value QZ_{LC}, where Q is a positive real number and Z_{LC} is an impedance (actually a resistance) which we will soon define. The gain of the amplifier will be, of course, $G = -Z_f/R_i$. The parallel feedback elements combine to create an impedance of:

$$\frac{1}{Z_f} = \frac{1}{j\omega L} + j\omega C + \frac{1}{QZ_{LC}}$$

At angular frequency ω_0, where $\omega_0^{\,2} = 1/LC$, the two imaginary terms in the expression for Z_f cancel, and at that frequency just the real-valued resistance remains, so $Z_f = QZ_{LC}$. This one frequency ω_0 at which Z_f is real is called the *resonant frequency* of the LC combination. Although the impedances of the L and C cancel at ω_0, neither one is equal to 0; in fact, the magnitude of each is equal to what is called the *characteristic impedance* of the LC combination, $Z_{LC} = \sqrt{L/C}$. The "*quality factor*" Q in this case is defined to be the ratio of the resistance in parallel with the L and C to their characteristic impedance Z_{LC}.

In terms of the LC resonant frequency ω_0 and characteristic impedance Z_{LC} the gain of the inverting amplifier in Figure 5-7 may be expressed as:

5.4
$$G = \frac{-Z_{LC}/R_i}{j\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right) + \dfrac{1}{Q}} \;; \quad Z_{LC} = \sqrt{\frac{L}{C}} \;, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Figure 5-8 shows Bode plots of the magnitude and phase of equation (5.4) when $Q = 50$. If $Q > 1$ then far from the resonant frequency Z_f is dominated by either the impedance of the L or the C, and the asymptotic responses (shown by the diagonal dashed lines in Figure 5-8) intersect at ω_0 with gain $G_0 \equiv |G(\omega_0)| = Z_{LC}/R_i$. The phases of the two asymptotes have opposite signs, however, and at ω_0 their contributions to the gain cancel, leaving only the parallel resistance to determine the gain. Thus the gain magnitude and phase change very

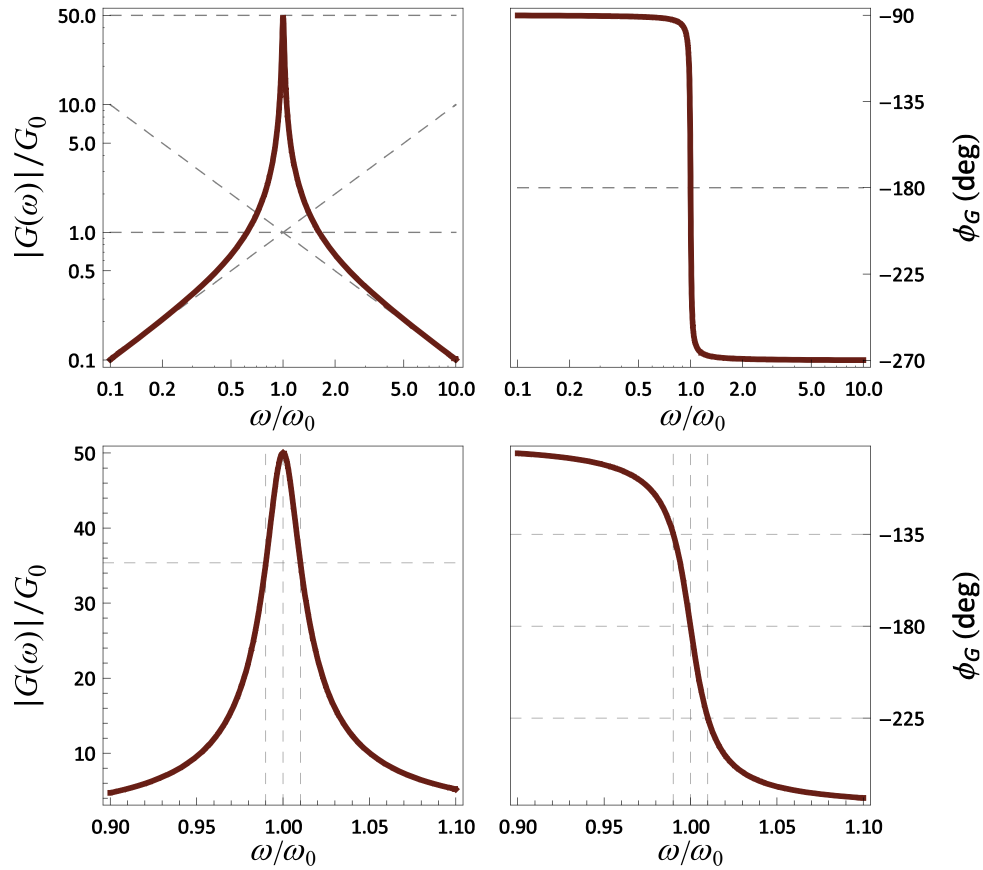


Figure 5-8: Bode plots of the *LC* resonant amplifier of Figure 5-7 for Q = 50. The gain magnitude is normalized to $G_0 = Z_{LC}/R_i$. Note that the asymptotic gain lines intersect at ω_0 with $G = G_0$, but the actual gain at ω_0 is Q times larger. Note that the phase plots have been "unwrapped" so that they remain continuous as the phase passes through −180° at resonance.

The lower plots show details of the response near ω_0. If Q is large, then the −3dB gain points are separated by $\Delta\omega_{-3dB}/\omega_0 = 1/Q$ (±1% around ω_0 in this example); the phase changes by 90° in this same small frequency range.

rapidly near ω_0, where the denominator of the gain function in expression (5.4) changes by a factor of Q within a narrow range of frequency of order ω_0/Q (lower pair of plots in Figure 5-8). This behavior is characteristic of the phenomenon of *resonance*. Because the denominator of (5.4) is quadratic in the angular frequency ω, the circuit in Figure 5-7 is an example of a *second-order system*.

Transient response of a high-Q resonant circuit

The gain of a circuit is the ratio of its output and input voltages, so (5.4) can be rewritten as

$$\left(\frac{j\omega}{\omega_0}\right)^2 V_{out} + \frac{1}{Q}\frac{j\omega}{\omega_0} V_{out} + V_{out} = \left(-\frac{j\omega}{\omega_0}\frac{Z_{LC}}{R_i}\right) V_{in}$$

This frequency-domain expression may be transformed to the time-domain by replacing $j\omega$ with a time derivative, so the gain expression (5.4) corresponds to the second-order, linear differential equation

5.5
$$\frac{1}{\omega_0^{\,2}}\ddot{v}_{out} + \frac{1}{Q\omega_0}\dot{v}_{out} + v_{out} = \frac{-Z_{LC}}{R_i}\frac{1}{\omega_0}\dot{v}_{in} = f(t)$$

If $Q > 1/2$, the homogeneous solution of (5.5) takes the form of a damped harmonic oscillation:

5.6
$$V_{out}(t) \propto e^{-\omega_0 t/2Q}\cos(\omega_T\, t + \phi); \qquad \omega_T = \omega_0\sqrt{1 - 1/(4Q^2)}$$

This *ringing* following a sudden transient input will persist for many cycles if Q is large; after Q cycles the ringing amplitude will still be about 4% of its starting value (see Figure 5-9).[1]

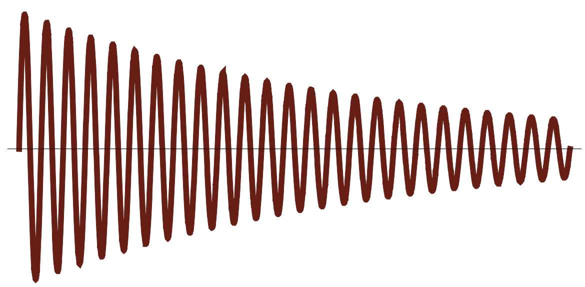


Figure 5-9: First 25 cycles of the output ringing of the circuit in Figure 5-7 for Q = 50. A negative step in the input would excite this response; the ringing initial amplitude would be approximately Z_{LC}/R_i times the input step amplitude.

High-Q resonant circuits such as this are useful to selectively amplify a very specific, narrow band of frequencies, such as when a radio receiver is tuned to a particular transmitter's frequency; for this reason they are sometimes called *tuned circuits*. Note that changing the input or the resonant frequency of the circuit must be followed by a waiting period of several Q's of cycles for the circuit's ringing in response to the change to largely dissipate — high-Q tuned circuits have a long *settling time*.

Note that if a resonant circuit's Q were infinite, the ringing would persist indefinitely — we would have a sine-wave oscillator. This feat may be accomplished by a judicious use of positive feedback in our linear, resonant system to maintain the sinusoidal output. Sine-wave oscillators are discussed in the section starting on page 5-22.

[1] The transient solution (5.6) could also have been found directly from the complex-valued, frequency-domain gain expression (5.4) first by substituting $s = j\omega$ into its denominator and then finding the two complex values for s which make the denominator vanish (the "poles" of the gain expression). The real part of s would give the rate of exponential decay, the imaginary part the frequency of the damped oscillations.

INTRODUCTION TO SECOND-ORDER ACTIVE FILTERS

Resonant circuits as filters

High-Q resonant circuits with responses similar to that shown in Figure 5-8 on page 5-9 are useful for the narrow-band, high gain filters needed in radio-frequency tuning circuits. For many applications, however, one needs low-pass and high-pass filters with a relatively flat pass-band response and a rapid reduction in gain (*roll-off*) as the signal frequency moves beyond the filter's cutoff frequency. A second-order resonant circuit with a modest Q ($Q \lesssim 1$) provides a solution which offers a flatter frequency response in the pass-band, a more dramatic out-of-band roll-off, and a faster settling time than you can achieve using simple, first-order RC filters. As a set of simple examples reminiscent of our first-order RC filters, consider the series LCR circuits configured as voltage dividers in various ways shown in Figure 5-10.

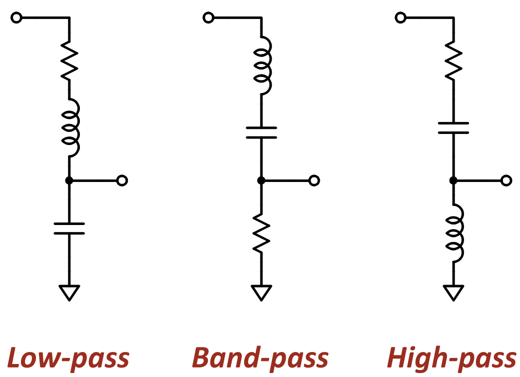


Figure 5-10: Series *LCR* circuits configured as simple filters. By forming a voltage divider and taking the output from the voltage across each of the various elements, various filter responses may be achieved.

For each of the LCR circuits above, the transfer function will have a denominator (D) like that of the gain function in equation (5.4), namely:

5.7
$$D \equiv j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) + \frac{1}{Q}; \qquad \omega_0 = \frac{1}{\sqrt{LC}}, \qquad Q = \frac{\sqrt{L/C}}{R}$$

These are called *second-order* filters because $D(\omega)$ is quadratic. At the resonant frequency ω_0 each circuit will have $1/D(\omega_0) = Q$. The numerators for the various LCR filters depend on the bottom element used in the voltage divider: each numerator is given by $Z(\omega)/Z_{LC}$, where $Z(\omega)$ is the impedance of the bottom element and $Z_{LC} \equiv \sqrt{L/C}$. The resulting transfer functions of these three filters for $Q = 1/\sqrt{2}$ (making them examples of *Butterworth* filters, described further in the next section) are plotted in Figure 5-11.

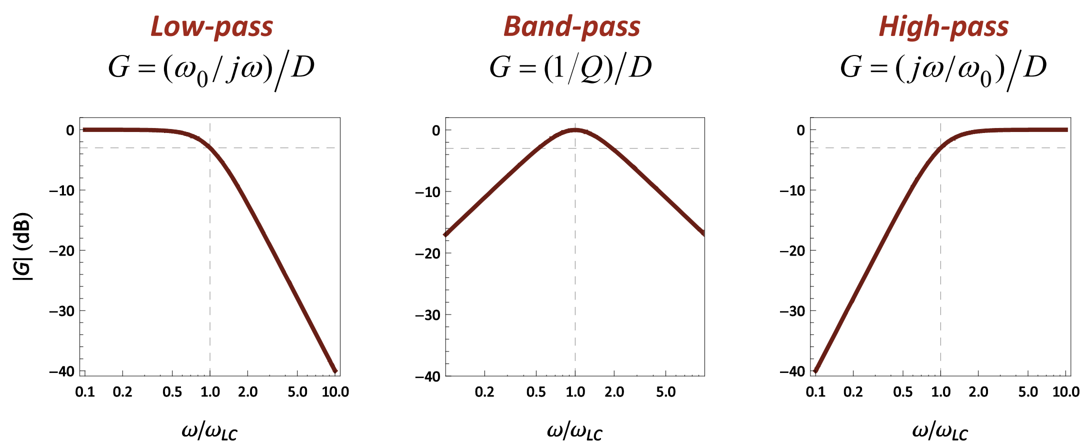


Figure 5-11: Transfer functions and Bode gain plots for second-order *Butterworth* filters. The −3dB gain points for the band-pass filter are at $0.52\omega_0$ and $1.93\omega_0$; for the others it is at ω_0. The out-of-band filter slopes are ω^{-2}, $\omega^{\pm 1}$, and ω^{+2}. The denominator D in the gain formulas is defined in equation 5.7.

Filter types: choosing the right Q

If one were to simply cascade two first-order *RC* low-pass filters with the same −3dB corner frequency (the filters separated by a voltage follower), then their combined response would be that of a second-order low-pass filter with $Q = 1/2$. Such a response is characteristic of a *critically damped* second-order system, and this behavior is often sought for by control system engineers, especially for mechanical systems such as automobile suspensions. For the purposes of electronic signal filtering, however, such a low Q system may sacrifice a bit too much in the way of frequency response, because its attenuation (*roll-off*) near its −3dB *cutoff frequency*[2] is quite gradual (*soft*), and the filter introduces significant phase shifts at frequencies far from this frequency. A high-Q response, on the other hand, may have a steep roll-off at its −3dB frequency, but it will exhibit severe gain peaking near its resonant frequency and will show a lot of ringing in response to a transient input. Making a choice of Q which results in the best compromise of gain flatness and phase shift in the pass-band, steepness of the roll-off at the −3dB frequency, and transient response is often a difficult one and will depend on your specific application. In this section we give examples of some popular choices designers typically consider.

The most popular choice for a second-order filter is probably the *Butterworth filter*, named after the British physicist Stephen Butterworth and characterized by $Q = 1/\sqrt{2}$ (as with the filters in Figure 5-11). Its low-pass filter version is called *maximally-flat* because it the highest Q second-order low-pass filter with a monotonically decreasing gain as frequency

[2] The −3dB *cutoff frequency* of a second-order filter is generally not its "corner" frequency. The second-order equivalent of a first-order *RC* filter's *corner frequency* would be its resonant frequency ω_0. In the case of a critically damped low-pass filter (Q = ½), the attenuation at this frequency would be its Q, or −6dB.

increases (i.e. no gain peaking or pass-band gain *ripple*). Its −3dB cutoff frequency is also its resonant frequency, ω_0.

A less common choice is the *Bessel filter*, with $Q = 1/\sqrt{3}$. Its lower Q places its −3dB cutoff frequency at $0.79\,\omega_0$, and the filter roll-off near its −3dB frequency is softer than that of the Butterworth (see left-hand graph in Figure 5-12 below). The advantage of the Bessel filter for certain applications, however, is that it has constant *group delay* for frequencies well within its pass-band, implying that it will cause the least distortion to the shape of a complicated waveform.

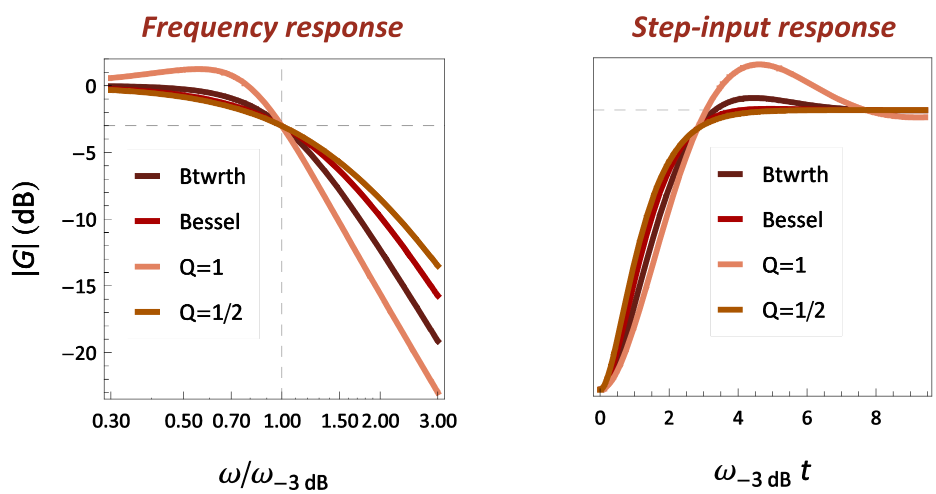


Figure 5-12: Frequency and transient responses of various 2nd order low-pass filters. All filters have the same −3dB frequency, but have different Q's (the Butterworth filter has legend "Btwrth"). The critically damped filter has Q = 1/2. The Q = 1 filter has mild gain peaking (+1.25 dB), but significant step response overshoot; its resonant frequency is at 0.79 of its −3dB frequency.

The *critically damped* filter mentioned previously ($Q = 1/2$) has its −3dB cutoff frequency at only $0.64\,\omega_0$, and the filter roll-off there is very gradual (soft). It has the distinction, however, of having the highest Q for which its transient response to a step input has no overshoot; its output settles following a step input more rapidly than for any other second-order filter with the same −3dB cutoff frequency (right-hand graph in Figure 5-12), although its resonant frequency must be made nearly 60% higher than that of an equivalent Butterworth filter.

A very simple VCVS active filter

The main drawback of *LCR* filters such as in Figure 5-10 is that they each require a high-quality inductor as one of the filter elements. This may not be a problem for a filter designed to operate at several MHz, but for frequencies below a few MHz your choice of suitable inductors may be limited (and such inductors are relatively expensive). Consequently, an *active filter* design implemented using op-amps with *RC* feedback networks is the more

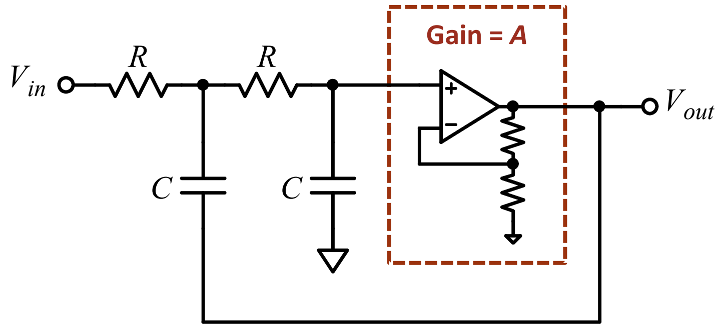


Figure 5-13: A simple version of the *VCVS* second-order, active low-pass filter, in which both the resistors and the capacitors of the two cascaded *RC* filters are chosen to be equal. The non-inverting op-amp amplifier stage has gain *A* (if a simple voltage follower is used, so *A* = 1, then the circuit would be called a *Sallen-Key* filter). The circuit's resonant frequency is $\omega_0 = 1/RC$, and its $Q = 1/(3-A)$. The circuit's in-band gain is *A*.

practical solution for a low-frequency filter. A gyrator circuit (Figure 5-6) may be used to emulate the inductor, but only if the circuit has one terminal of the inductor connected to ground (only the high-pass series *LCR* filter discussed previously meets this requirement). Fortunately, electronics designers have invented several op-amp circuits employing both positive and negative feedback which implement quite effective second-order filters. In this section we discuss one of these, called a *Voltage-Controlled Voltage Source* (*VCVS*) active filter.

A simple *VCVS* circuit for a second-order low-pass filter is shown in Figure 5-13. It is clearly a cascaded pair of simple *RC* filters followed by a noninverting op-amp gain stage. The wrinkle to this circuit, however, is that the first *RC* filter stage is terminated not by grounding its capacitor but by connecting it to the amplifier output, providing some positive feedback (also called *bootstrapping*). This clever change makes the filter's Q as well as its resonant frequency ω_0 adjustable by selecting appropriate component values and amplifier gain A.

Rather than analyze the general case where the component values and amplifier gain are all arbitrary, we consider only the special case wherein both resistors and both capacitors are chosen to have equal values R and C. In this special case the transfer function of the circuit is:

Simplified *VCVS* low-pass filter (Figure 5-13)

5.8
$$G = \frac{A}{1 - \dfrac{\omega^2}{\omega_0^{\,2}} + j\dfrac{\omega}{Q\omega_0}}; \qquad \omega_0 = \frac{1}{RC}, \quad Q = \frac{1}{3-A}$$

This result is derived in the supplementary information section starting on page 5-25. The circuit is called a *VCVS* filter because the op-amp stage acts as a *voltage-controlled voltage source* (which is just another way of saying that its output voltage is a function of its input voltage, and that its output impedance is small). Examining the circuit should make it clear that its gain at very low frequency, in which case the capacitors' impedances get very large, should approach that of the noninverting amplifier stage, A. Not as clear, perhaps, is that the positive feedback stability criterion on page 5-4 will demand that $A < 3$ (note from the equations (5.8) that $Q \to \infty$ as $A \to 3$). If the resistor and capacitor values are not well-matched, then the circuit's ω_0 and Q will vary a bit from the expressions in 5.8.

High-pass and band-pass versions of the simple *VCVS* filter are shown in Figure 5-14. If $A = 1$ (i.e. a voltage follower is used for the noninverting op-amp stage), then the circuit is called a *Sallen-Key filter* after its inventors at the MIT Lincoln Laboratory in 1955. The section *Sallen-Key low-pass filter* on page 5-20 shows how to design such a filter.

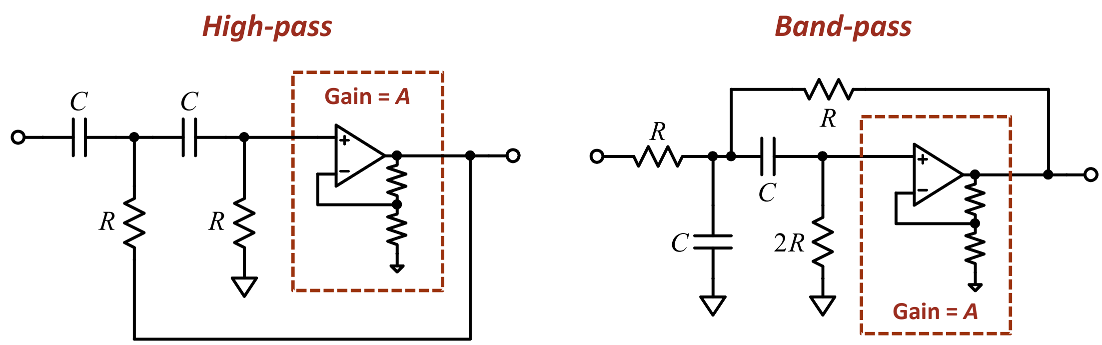


Figure 5-14: Other *VCVS* second-order filters. As with the low-pass filter, the filters' ω_0 = 1/*RC*, and Q = 1/(3–*A*). The in-band gain of the high-pass filter is *A*; the band-pass filter's gain at ω_0 = *QA*. Note that one resistor in the band-pass version of the filter has value 2*R*.

Other active filter circuit topologies; the state-variable filter

Another common second-order active filter which uses only a single op-amp is the *multiple-feedback* (*MFB*) design, which is most useful for high-Q or high-gain filter stages. A low-pass version of the circuit is shown in Figure 5-15; to get a high-pass design, simply swap resistors for capacitors and vice versa in the circuit. Note that this is an inverting amplifier (the pass-band gain is $-R_2/R_1$) and that the op-amp is configured as an integrator. We won't discuss this filter design any further, but you can find several references for designing this type of filter in the lab library or on the web.

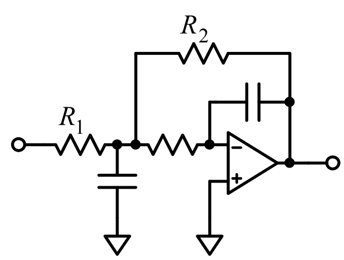


Figure 5-15: *Multiple-feedback* low-pass filter circuit.

A more flexible design than either the *VCVS* or *MFB* topologies is the second-order *state-variable filter* shown in Figure 5-16, which is from the Texas Instruments ASLK Pro Manual, page 32; that document's *Experiment 5* shows how to convert this circuit to a voltage-controlled filter. Although this filter uses 4 op-amps, it simultaneously provides low-pass, band-pass, high-pass, and even *band-stop* filter outputs, and both its pass-band gain A and its Q are easily adjustable. Note that the circuit topology is formed from a cascade of two inverting integrators and a gain -1 inverting, summing amplifier whose output is fed back to become the input to the first integrator. To this basic loop is added a *feed-forward* section around the middle integrator, and in this section the input signal is summed with the output of the first integrator.

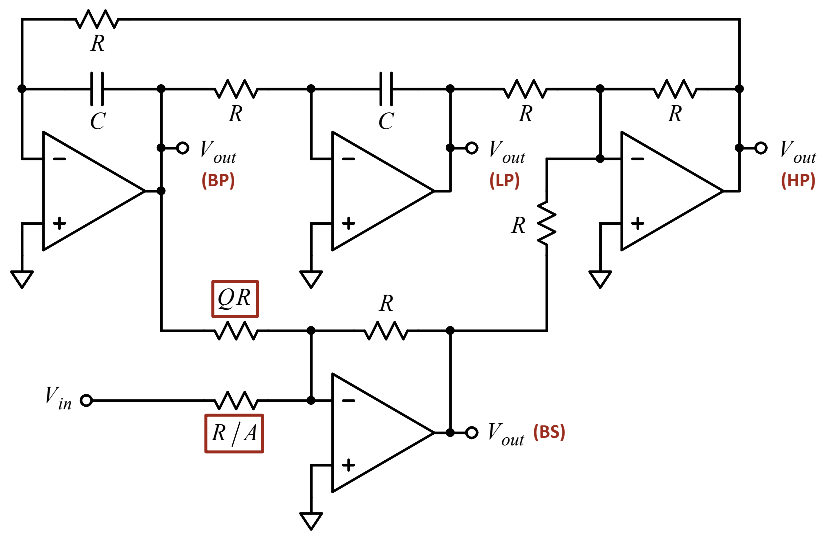


Figure 5-16: A *Universal Active Filter* (*UAF*), a form of second-order, state-variable active filter. This design is from the Texas Instruments ASLK Pro Manual, ©Texas Instruments, 2012. It simultaneously provides high-pass (HP), low-pass (LP), band-pass (BP), and band-stop (BS) outputs. The pass-band filter gain is A; its resonant frequency is $\omega_0 = 1/RC$.

The resonant frequency is, naturally, $\omega_0 = 1/RC$. The gain in the pass-band of the low-pass, high-pass, and band-stop filters is A, which from Figure 5-16 is just the gain for the input signal of the summing amplifier at the bottom of the figure; the gain of the band-pass filter at ω_0 is QA. From the circuit schematic you can see that Q is simply the reciprocal of the gain of the feed-forward signal from the first integrator through this same summing amplifier. The response of the band-stop filter is shown at right; its output phase changes by 360° across the band-stop resonance.

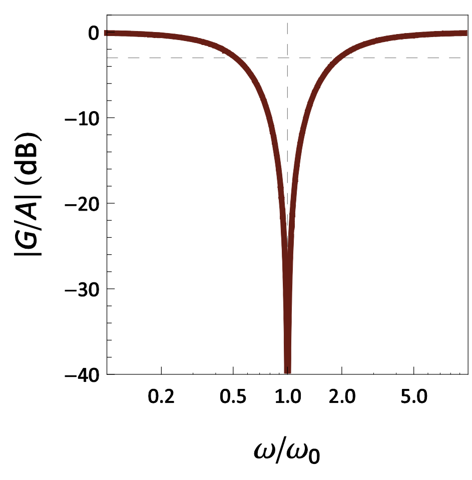


Figure 5-17: *Band stop* filter response.

You can see how the state-variable filter works by first reviewing Figure 5-11 on page 5-12. Starting with the high-pass filter's response function in that figure, note that you can obtain the response functions of the band-pass and low-pass filters by successive multiplications by $\omega_0/j\omega$. But this is the same as integrating in the time-domain, so clearly the successive integrators in the state-variable filter loop (Figure 5-16) convert the high-pass output to band-pass and then low-pass outputs, as long as $\omega_0 = 1/RC$. Now with a little bit of arithmetic and keeping in mind that the amplifiers are all inverting, you should be able to show that the two summing amplifiers combine the low-pass, band-pass, and input signals in just the right way to generate the high-pass response.

Note that the input impedance of the filter circuit is equal to the single input resistor value R/A, so you may need to add a voltage follower to the input if you require a large input impedance.

If you need to be able to easily and independently adjust the gain or Q of a second-order active filter or if you need multiple filter outputs to split an input signal into frequency sections, then the state-variable filter is a good choice.

> The state-variable filter is really only suitable for filters with a modest Q of 2 or less (which is by far the most commonly encountered requirement); otherwise extremely good component matching of the various resistors and capacitors may be necessary, especially for the band-stop filter output.

Prelab exercises

1. Consider the voltage divider with a negative impedance circuit as one "element" shown in Figure 5-18 at right.

   (1) What is the gain (V_{out}/V_{in})?

   (2) Which way should the op-amp's inputs be oriented (+*Input* at top or –*Input* at top)? (Assume that the source of V_{in} is a perfect voltage source and the load at V_{out} has infinite input impedance.)

   (3) What are the circuit's input and output impedances? (assume that the source of V_{in} is a perfect voltage source and the load at V_{out} has infinite input impedance).

   (4) If $V_{out} = 1.0\text{V}$, then what is the voltage at the op-amp output?

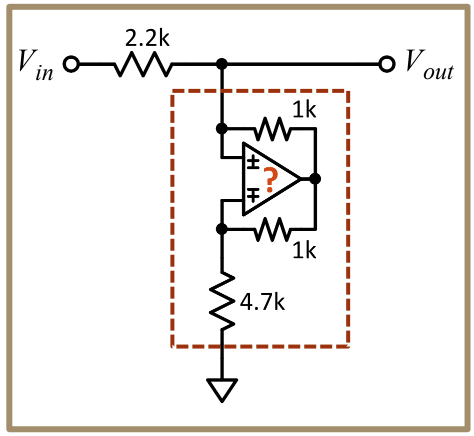


Figure 5-18: A voltage divider with a negative impedance (boxed circuit).

2. Consider the *Howland current pump* shown in Figure 5-2 on page 5-2. What is the op-amp output voltage V_{out} in terms of the input and load voltages V_{in-}, V_{in+}, and V_{Load}?

3. Sketch the phase response Bode plots to go with each of the gain magnitude plots shown in Figure 5-11 on page 5-12 for the *LCR* filters shown in Figure 5-10.

4. Consider the simple *VCVS* low-pass filter in Figure 5-13 on page 5-14. What should be the ratio of the two resistors of the noninverting amplifier (i.e. R_f/R_i) if you want a Butterworth filter response?

5. Design a cascaded amplifier-filter circuit which has a pass-band gain = 10 and has a Butterworth low-pass filter response with a –3dB cutoff frequency of 16 kHz. Use a *VCVS* circuit for the low-pass filter (Figure 5-13 on page 5-14).

   Circuit specifications:

   Input impedance: 1 MegΩ
   Pass-band gain: 10 (noninverting)
   Filter spec: Butterworth low-pass, –3dB frequency of 16 kHz

   When you cascade the amplifier and filter, should you put the amplifier or the filter first? Which of the above specifications will most strongly influence your choice?

   Hints for resistor selections: $33\text{k}/56\text{k} = 0.59$; $27\text{k}/5.1\text{k} = 5.3$.

LAB PROCEDURE

Overview

During lab you will construct the amplifier-filter you designed as part of the prelab exercises. Next you will investigate the capabilities of the state-variable filter architecture.

Amplifier-filter using the VCVS architecture

Construct the amplifier-filter circuit you designed (Prelab exercise 5) in the white breadboard area using a supplied TL082 dual op-amp IC (refer to the TL082 data sheet, page 3 for the IC pin-out — *make sure you look at the pin-out for the correct IC!*).

Use the *Frequency Response* application to plot the frequency response of your circuit. Next, input a square wave and use the oscilloscope to investigate the filter's transient response to a step input. Compare your results to the graphs in Figure 5-12 on page 5-13.

State-variable (UAF) filter

Using an additional TL082 IC, now construct a *UAF* with a 16kHz resonant frequency as in Figure 5-16 on page 5-16. Initially choose a Q of 1 and a pass-band gain A of 1 for component selections. You should be able to reuse the same *RC* pairs you used for the *VCVS* filter for the two integrator sections of the circuit. Check the frequency responses of each of the filter's four outputs. Check the transient response of the low-pass filter output.

Now, by changing the value of the appropriate resistor, increase the Q to approximately 10 and look again at the outputs' frequency responses, especially that of the band-stop (notch) filter. Check the low-pass filter's transient response.

Additional Circuits

If you have time, construct a circuit from the **MORE CIRCUIT IDEAS** section, from an earlier experiment, or one of you own design; the *Wien bridge oscillator* shown in Figure 5-22 on page 5-24 is an interesting choice. Another choice is to construct the amplifier with a negative impedance circuit shown in Figure 5-18 using your solution to Prelab Exercise 1 as a guide. Measure its gain and compare with your calculations.

Lab results write-up

As always, include a sketch of the schematic with component values for each circuit you investigate, along with appropriate oscilloscope screen shots.

More circuit ideas

Sallen-Key low-pass filter

The simplified VCVS filter presented earlier (Figure 5-13 on page 5-14) with equal Rs and Cs in the cascaded low-pass filters is rather inflexible, since the gain of the amplifier must be chosen to give the required Q. It is often desirable for the filter's pass-band gain to equal 1, so that the amplifier configuration used is just a simple voltage follower. In this case one gets the standard *Sallen-Key low-pass filter* shown in Figure 5-19. The resistors and capacitors now generally need to have unequal values as indicated in the figure by the ratios ρ and κ.

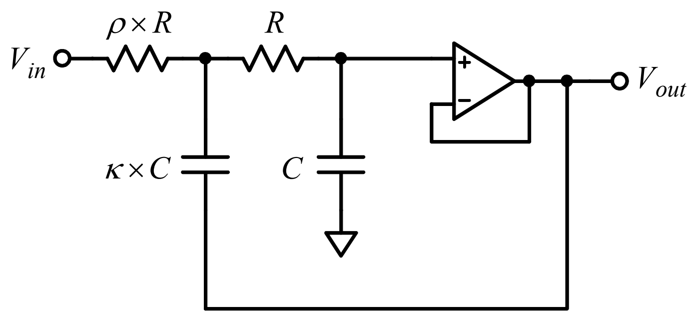


Figure 5-19: Sallen-Key low-pass filter. By using a voltage follower for the *VCVS* amplifier, the pass-band gain of the filter is unity; the ratios ρ of the resistors and κ of the two capacitors are chosen to select the filter's Q and resonant frequency ω_0.

The relationships between the filter's ω_0 and Q and these component value ratios are given by:

5.9
$$\omega_0 = \frac{1}{RC\sqrt{\kappa\rho}} \qquad Q = \frac{\sqrt{\kappa\rho}}{\rho+1}$$

Because the selection of capacitor values is generally more limited than that for resistors, one usually picks a convenient capacitor ratio such as $\kappa = 10$. With this choice the resistor value ratio is determined from the required Q:

$$\rho = (5/Q^2 - 1) \pm \sqrt{(5/Q^2 - 1)^2 - 1}$$

Clearly, the resistor value ratio ρ must be real and positive, so the argument of the square root must be nonnegative; thus the choice $\kappa = 10$ will be valid only for $Q \le \sqrt{5/2}$. This requirement is easily met by any reasonable filter Q (usually $Q < 1$). For a Butterworth filter response $Q = 1/\sqrt{2}$, and:

Sallen-Key Butterworth filter component values

5.10
$$\kappa = 10, \quad \rho = 9 \pm \sqrt{80} \approx 18 \text{ or } \boxed{1/18} \quad \rightarrow \quad \omega_0 \approx \frac{1}{(0.75)RC}$$

By choosing $\rho = 1/18$, the filter's corner frequency is reasonably close to $1/RC$, and the impedance of the second RC filter is quite a bit larger than that of the first, so the first RC filter's response is only marginally affected by the load of the second RC filter. Once you have chosen appropriate values for the ratios κ and ρ, choose C and R to give the desired ω_0.

Many tools are available to help you design active filters. My favorite is this web-based application available from Analog Devices:

Analog Filter Wizard
http://www.analog.com/designtools/en/filterwizard/

A simple nonlinear amplifier

In the next section we discuss the design of sine-wave oscillators, which are essentially resonant circuits with a Q of "exactly" the point at complex ∞ when the resonator output amplitude takes on a certain target value; if its amplitude is too large, then the Q is finite and positive so that the output amplitude decays back to the target value, whereas if the amplitude is too small (as when the circuit is first turned on), then the Q becomes *negative*, so the amplitude *increases* toward the target value.

This seemingly remarkable behavior may be accomplished by using a simple *nonlinear amplifier whose gain decreases as its output amplitude rises*. A noninverting version of such an amplifier is shown in Figure 5-20.

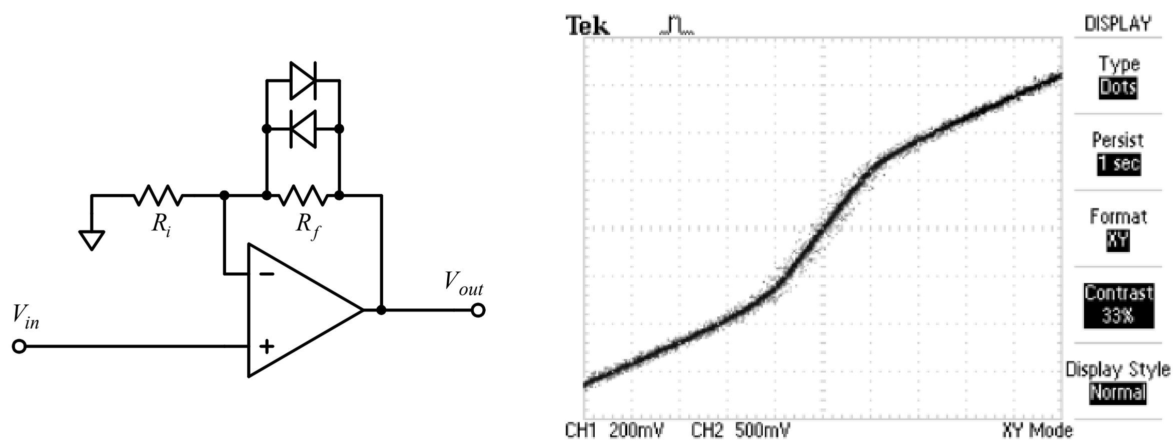


Figure 5-20: A simple nonlinear amplifier. For large signal inputs the amplifier's *dynamic gain* drops to approximately 1, as explained in the text and as illustrated by the V_{out} v. V_{in} data shown in the right-hand image (R_i = 1k, R_f = 2.2k for the data shown).

Here's how the circuit works:

If the feedback current (V_{in}/R_i) is small, then the amplifier's gain is given by the normal noninverting amplifier formula $1 + R_f/R_i$. As the signals get larger, the feedback current goes up; when the voltage drop across R_f approaches 0.6 V, the silicon diodes begin to conduct significant current, reducing the effective feedback resistance and thus reducing the amplifier's gain. Since the diodes will limit the voltage drop across R_f to about 0.6 V as the input signal continues to rise, we see that for large inputs V_{out} will remain a diode drop above V_{in}, and the amplifier's *dynamic gain* (dV_{out}/dV_{in}) is reduced to ≈ 1. This behavior is illustrated by the V_{out} vs. V_{in} data plotted in Figure 5-20, for $R_i = 1.0\,\text{k}$ and $R_f = 2.2\,\text{k}$. When the input voltage is small, the amplifier's gain is 3.2, as indicated by the slope of the curve near the origin. At an input voltage of $\approx 0.25\text{V}$, however, the slope of the curve changes abruptly as the diodes begin to conduct (the voltage across R_f is then ~0.6V). As V_{in} continues to increase, $V_{out}/V_{in} \to 1$ ($V_{out}/V_{in} \approx 1.6$ for $V_{in} = 1\text{V}$).

Sine-wave oscillators

As mentioned in the section on resonant circuits, if a second-order system's $Q \to \infty$, then its transient response ringing will continue indefinitely at its resonant frequency: it has become a sine-wave oscillator. In this section we briefly present a couple of simple sine-wave oscillator circuits. As a first example, consider the LC resonant oscillator in Figure 5-21.

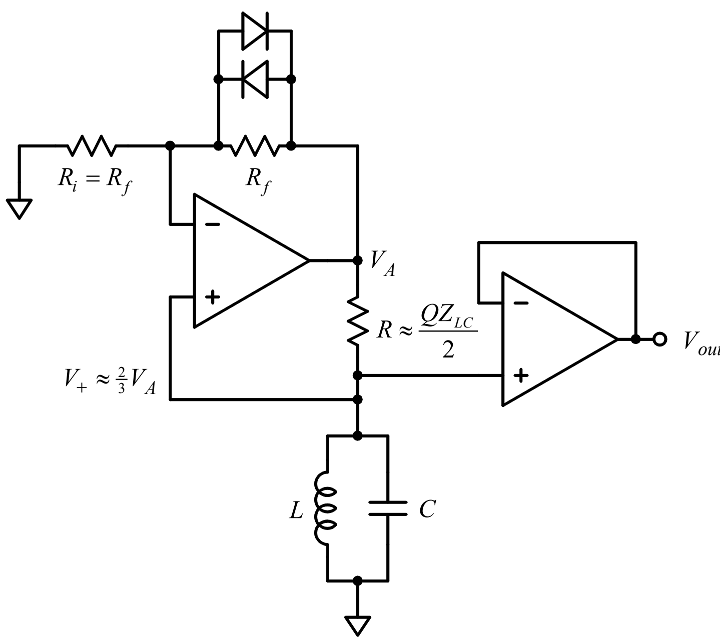


Figure 5-21: A simple *LC* resonant sine-wave oscillator. Its operation is explained in the text.

The oscillator output is taken (via the voltage follower) from the signal across the *LC* resonator, producing a low-distortion sine-wave output, even though the output of the nonlinear amplifier may be quite distorted.

Some positive feedback is required to sustain the circuit's oscillation; in this circuit it is generated by the voltage divider consisting of the resistor R and the parallel LC resonator. This feedback recirculates a fraction of the output of a noninverting nonlinear amplifier back to its input. At the resonant frequency $\omega_0 = 1/\sqrt{LC}$ the impedance of the LC resonator becomes very large: $QZ_{LC} = Q\sqrt{L/C}$; for a good-quality resonator, we would expect Q~150 or more, and the LC component values should be chosen so that $Z_{LC} \sim$ 1k to 10k. By choosing R to be about 1/2 of this resonant impedance, the positive feedback fraction is about 2/3 at frequency ω_0. At frequencies more than a couple ω_0/Q away from resonance, on the other hand, the *LC* impedance is $\sim \pm jZ_{LC}$ or less, so positive feedback at any other frequency is $\ll 1$ and has a 90° phase shift.

For sustained, constant amplitude oscillation, the gain around the feedback loop, including the amplifier, must be exactly 1. The nonlinear amplifier's gain for a small input signal is 2 when $R_f = R_i$, so for a small signal at ω_0 the loop gain is approximately $\frac{2}{3} \times 2 > 1$. Thus a small ω_0 signal will grow in amplitude exponentially quickly, whereas a signal at any other frequency will die away. As the ω_0 signal's amplitude grows, the nonlinear amplifier's gain will decrease until the amplitude is reached such that the loop gain is exactly unity, and the nonlinear amplifier will stabilize its output amplitude at that value. If the feedback resistors are each 1k, then the circuit's output amplitude will be $\sim$1V peak.

Because the diodes act to clamp the amplifier's peak output, the amplifier's output will take the form of a distorted sinusoid. The filtering action of the resonant *RLC* voltage divider, however, will provide a much more pure sinusoid across the *LC* pair, so that is where the oscillator's output is taken (using a voltage follower to isolate the resonator from its load). The distortion is minimized when the small-signal loop gain around the nonlinear amplifier at frequency ω_0 just barely exceeds 1, so trimming of the value of *R* may be necessary if really low distortion is required. Unfortunately, if the small-signal loop gain were to drop even a tiny bit below 1, then the circuit will cease to oscillate. If slightly higher distortion in the sine-wave output is tolerable, then the diodes in Figure 5-21 may be unnecessary: saturation of the amplifier op-amp output (V_A in Figure 5-21) will limit the oscillator's amplitude, but the signal at the *LC* resonator (where the output is taken) will still be a quite nice sinusoid.

Very similar in concept to the *LC* oscillator presented above, the *Wien bridge oscillator* uses *RC* pairs to form its resonant voltage divider as shown in Figure 5-22 on page 5-24 (a Wien bridge-type oscillator was the first product offered by the Hewlett-Packard company, back in 1939).

The two *RC* pairs in the Wien bridge circuit form a resonant voltage divider to provide some positive feedback to sustain oscillation. The resonant frequency is $\omega_0 = 1/RC$, but the filter's *Q* is only $1/3$ when the *RC* pairs are perfectly matched. The divider ratio is also $1/3$ at ω_0, so the nonlinear amplifier must have a small-signal gain of at least 3 for the circuit to oscillate. The diode clamping will severely distort the oscillator output if the amplifier's small-signal gain is even more than a few percent above 3, so some trimming of either R_f or R_i will be necessary to limit the distortion in the output waveform.

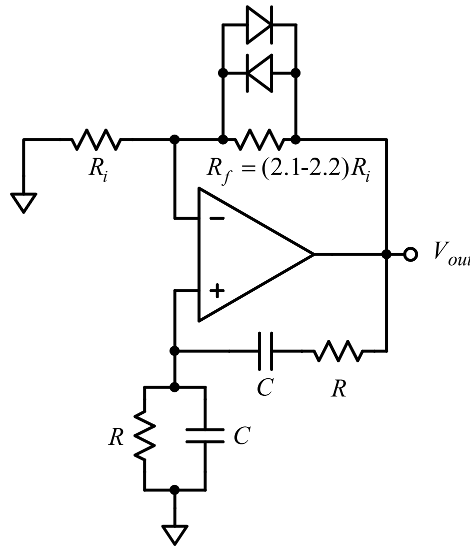


Figure 5-22: The *Wien bridge* oscillator uses two *RC* pairs to form its resonant voltage divider which provides positive feedback to sustain oscillation. The oscillation frequency is $\omega_0 = 1/RC$.

At ω_0 the positive feedback fraction is 1/3, so the small-signal gain of the nonlinear amplifier must be at least 3. The resonant divider's Q =1/3, so, because of the diode clamping, achieving a tolerable level of distortion will nearly always require trimming of the ratio R_f/R_i to just above 2.

An alternative (and much better) method of controlling the amplifier's gain is to use an incandescent light bulb (one with a glowing filament) in place of resistor R_i as shown at right; this is the method used in a truly low-distortion Wien bridge oscillator. This technique was invented by L. A. Meacham of Bell Laboratories in 1938 and was incorporated in the Hewlett-Packard product mentioned previously. As the bulb's filament heats up, its resistance increases, lowering the amplifier's gain. Because a change in the filament temperature is gradual (taking much longer than the oscillator's output period), it has essentially no effect on the output waveform shape. For our Wien bridge oscillator, the incandescent bulb used should have a resistance when cold of about 100Ω; the amplifier's feedback resistor should have a value about 2.5 times higher. With this version of the oscillator's amplifier, the output signal's distortion may be so low as to be very hard to measure.

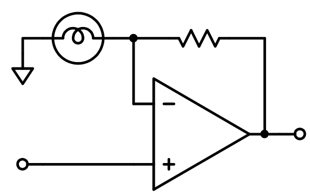


Figure 5-23: An amplifier whose gain is controlled by the temperature of the incandescent bulb used as the op-amp's R_i.

There are many other designs for op-amp sine-wave oscillators. The following article by Texas Instruments provides a nice summary and design procedures for several practical ones:

http://www.sophphx.caltech.edu/Physics_5/Useful_circuits/TI_Sine_Oscillators.pdf

Additional Information about the Text Ideas and Circuits

Derivation of the simple VCVS filter gain

The simple VCVS low-pass filter pictured in Figure 5-13 on page 5-14 has a transfer function given by equations 5.8, repeated below:

5.8
$$G = \frac{A}{1 - \dfrac{\omega^2}{\omega_0^{\,2}} + j\dfrac{\omega}{Q\omega_0}}; \qquad \omega_0 = \frac{1}{RC}, \qquad Q = \frac{1}{3-A}$$

In this section we derive this result. The circuit is shown again in the figure below, where relevant node voltages V_1 and V_2 are also indicated:

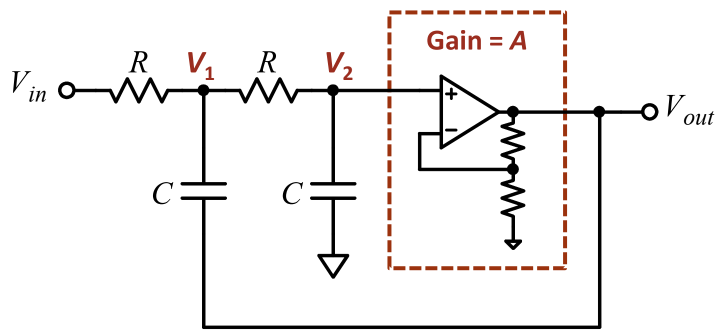


Figure 5-24: The simple version of the *VCVS* second-order, active low-pass filter, in which both the resistors and the capacitors of the two cascaded *RC* filters are chosen to be equal.

The op-amp sub-circuit in the dashed box is just a noninverting amplifier with a gain of A, so

$$V_{out} = AV_2 \quad \rightarrow \quad V_2 = V_{out}/A$$

The voltage V_2 is derived from V_1 using an RC voltage divider, so

$$V_2 = V_1 \frac{Z_C}{R + Z_C}$$

Combining the above equations:

$$V_2 = \frac{V_{out}}{A} = V_1 \frac{Z_C}{R + Z_C}; \qquad \therefore\ V_1 = \frac{V_{out}}{A}\left(1 + \frac{R}{Z_C}\right)$$

The voltage V_1 can also be found from the other three voltages using the generalized voltage divider equation derived in Experiment 1:

$$V_1 = \left(\frac{V_{in}}{R} + \frac{V_2}{R} + \frac{V_{out}}{Z_C}\right) \div \left(\frac{1}{R} + \frac{1}{R} + \frac{1}{Z_C}\right); \qquad V_2 = V_{out}/A$$

Equating the above two expressions for V_1 and substituting V_{out}/A for V_2:

$$\frac{V_{out}}{A}\left(1+\frac{R}{Z_C}\right) = \left(\frac{V_{in}}{R}+\frac{V_{out}}{AR}+\frac{V_{out}}{Z_C}\right)\div\left(\frac{1}{R}+\frac{1}{R}+\frac{1}{Z_C}\right)$$

$$\frac{V_{out}}{A}\left[\left(1+\frac{R}{Z_C}\right)\left(\frac{2}{R}+\frac{1}{Z_C}\right)-\left(\frac{1}{R}+\frac{A}{Z_C}\right)\right] = \frac{V_{in}}{R}$$

$$\frac{V_{out}}{A}\left(\frac{2}{R}+\frac{1}{Z_C}+\frac{2}{Z_C}-\frac{1}{R}-\frac{A}{Z_C}+\frac{R}{{Z_C}^2}\right) = \frac{V_{out}}{A}\left(\frac{1}{R}+\frac{3-A}{Z_C}+\frac{R}{{Z_C}^2}\right) = \frac{V_{in}}{R}$$

Now we're almost there. Rearranging the final expression above, with $Z_C = 1/j\omega C$:

$$\frac{V_{out}}{A}\left(\frac{1}{R}+\frac{3-A}{Z_C}+\frac{R}{{Z_C}^2}\right) = \frac{V_{in}}{R}$$

$$\frac{V_{out}}{V_{in}}\left(1+\frac{R^2}{{Z_C}^2}+\frac{R}{Z_C}(3-A)\right) = A$$

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{A}{1-(\omega RC)^2+j(3-A)(\omega RC)}}$$

Comparing this expression with equation 5.8, we see that they match if $\omega_0 = 1/RC$ and $Q = 1/(3-A)$.