Ph177 Exercise

Polarization 1

In order to fully understand the meaning of the two-photon "entangled" polarization state $|HH\rangle + |VV\rangle$, we should first go over the description and manipulation of the polarization state of a single photon (this should be a review of what you've learned in Ph12 and Ph125). We define the polarization of a photon state as a description of the oscillation of its electric field vector when viewed by an observer *looking back along its direction of propagation toward its source*. The normalized ket vectors $|V\rangle$ and $|H\rangle$ represent vertical and horizontal polarization states of the photon, respectively. These two kets form an orthonormal basis for the 2-dimensional Hilbert space of polarization state vectors; any arbitrary polarization may be represented by a normalized ket vector which is linear combination of these two (α and β may be complex numbers):

$$|\psi\rangle = \alpha|V\rangle + \beta|H\rangle; \quad |\alpha|^2 + |\beta|^2 = 1.$$

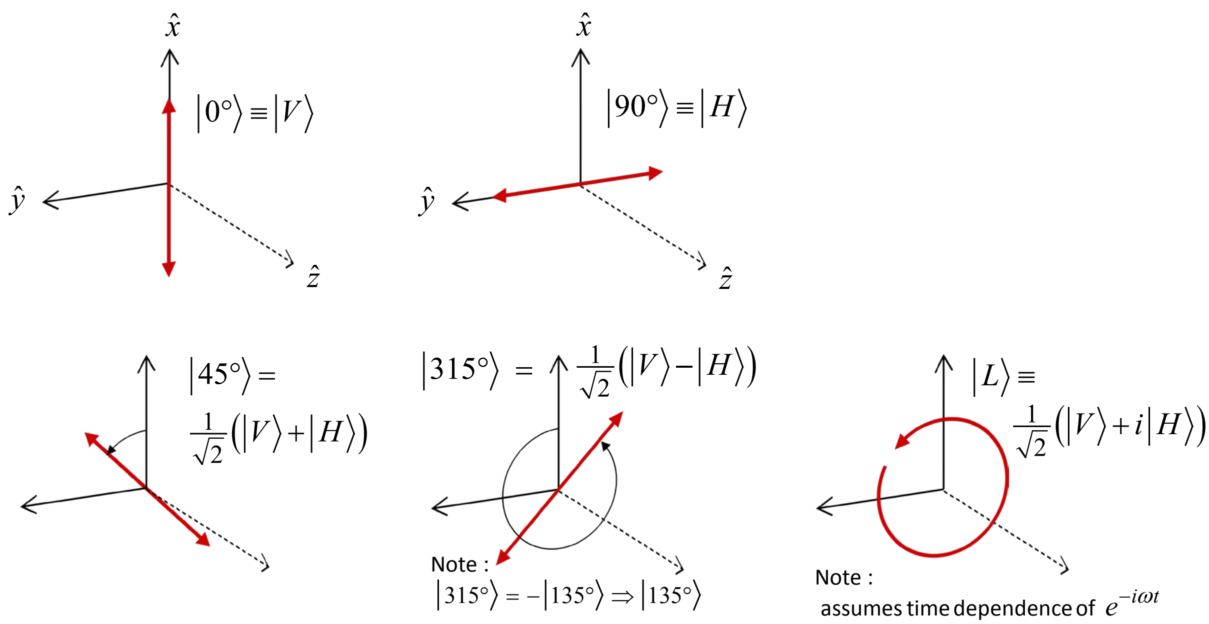
The coefficients α and β may be calculated in the usual way:

$$\alpha = \langle V|\psi\rangle, \quad \beta = \langle H|\psi\rangle; \quad \text{with} \quad \langle V|V\rangle = \langle H|H\rangle = 1, \quad \langle V|H\rangle = 0.$$

Of course, any normalized state ket may be multiplied by a complex constant with unit modulus, $e^{i\phi}$ (ϕ real), without affecting the state it represents (see Problem 3(b)). Our phase convention usually will be to choose the coefficient of $|V\rangle$ to be real and nonnegative. To the state represented by the ket vector $|\psi\rangle$ is a corresponding dual-space vector (bra):

$$\langle\psi| = \alpha^*\langle V| + \beta^*\langle H| = \langle\psi|V\rangle\langle V| + \langle\psi|H\rangle\langle H|.$$

Some examples are shown below. (in these figures, oscillations are in the x–y plane; propagation is in the $\hat{z}$ direction; rotation angle is measured counterclockwise from the vertical, $\hat{x}$).



The ket $\left|L\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|V\right\rangle + i\left|H\right\rangle\right)$ represents left-hand circular polarization; $\left|R\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|V\right\rangle - i\left|H\right\rangle\right)$ is the ket for right-hand circular polarization. These definitions assume a convention for the time-variation of the wave's complex phase of $e^{-i\omega t}$ (the "physicist's phase convention").

Interposing a linear polarizer oriented at some angle θ counterclockwise from the vertical may be represented as a Hermitian *projection operator* $\mathbf{P}(\theta)$ operating on the photon polarization state $\left|\Psi\right\rangle$. This operator has two eigenstates: the state $\left|\theta\right\rangle$ with eigenvalue 1 and the state $\left|\theta+90°\right\rangle$ with eigenvalue 0. Thus the linear polarizer's operator has the matrix representation:

$$\mathbf{P}(\theta) \;=\; \left|\theta\right\rangle 1 \left\langle\theta\right| \;+\; \left|\theta+90°\right\rangle 0 \left\langle\theta+90°\right| \;=\; \left|\theta\right\rangle\left\langle\theta\right|.$$

Clearly, $\mathbf{P}(0°) = \left|V\right\rangle\left\langle V\right|$ and $\mathbf{P}(90°) = \left|H\right\rangle\left\langle H\right|$. The probability for a photon with polarization state $\left|\Psi\right\rangle$ passing through a linear polarizer and then being observed in the state $\left|\theta\right\rangle$ is thus

$$\left|\mathbf{P}(\theta)\left|\Psi\right\rangle\right|^2 = \left\langle\Psi|\theta\right\rangle\left\langle\theta|\Psi\right\rangle.$$

Note that a $180°$ rotation of the polarizer doesn't change anything, and that the kets $\left|\theta\right\rangle$ and $\left|\theta \pm 180°\right\rangle$ represent the same polarization state, because they are proportional (differ by a factor of -1).

Problem 1:

(a) What are the representations of the states $\left|\theta\right\rangle$ and $\left|\theta+90°\right\rangle$ in the $\left|V\right\rangle, \left|H\right\rangle$ basis?

(b) Show that $\left|\mathbf{P}(\theta)\left|L\right\rangle\right|^2 \;=\; \left|\mathbf{P}(\theta)\left|R\right\rangle\right|^2 \;=\; 1/2$, for any angle θ.

(c) The states $\left|\theta\right\rangle$ and $\left|\theta+90°\right\rangle$ also form an orthonormal basis. Write $\left|V\right\rangle$ and $\left|H\right\rangle$ as linear combinations of $\left|\theta\right\rangle$ and $\left|\theta+90°\right\rangle$.

The *rotation operator* $\mathbf{R}(\theta)$ rotates a linear polarization state $\left|\phi\right\rangle$ counterclockwise by angle θ. In particular, $\mathbf{R}(\theta)\left|V\right\rangle = \left|\theta\right\rangle$, and $\mathbf{R}(\theta)\left|H\right\rangle = \left|\theta+90°\right\rangle$. Again, keep in mind that a $180°$ rotation just multiplies a polarization state's ket by -1, which is just a phase, so the polarization state doesn't change. Thus, for example, $\mathbf{R}(90°)\left|H\right\rangle = -\left|V\right\rangle \Rightarrow \left|V\right\rangle$. A *half-wave plate* is an optical element constructed from a *birefringent* material. It has a fast axis (oriented in the x–y plane) which may be rotated to any desired angle θ. Its action on a polarization state vector may be described by the operator $\mathbf{W}_{1/2}(\theta)$. The eigenvectors of $\mathbf{W}_{1/2}(\theta)$ are $\left|\theta\right\rangle$ with eigenvalue 1 and $\left|\theta+90°\right\rangle$ with eigenvalue –1, so the representation of a half-wave plate's action is:

$$\mathbf{W}_{1/2}(\theta) \;=\; \left|\theta\right\rangle\left\langle\theta\right| \;-\; \left|\theta+90°\right\rangle\left\langle\theta+90°\right|.$$

Problem 2:

(a) Find the eigenvectors and eigenvalues of the polarization rotation operator $\mathbf{R}(\theta)$. Write the representation of this operator in terms of its eigenvectors.

(b) Show that $\mathbf{W}_{1/2}(\theta)\left|V\right\rangle = \left|2\theta\right\rangle = \mathbf{R}(2\theta)\left|V\right\rangle$.

(c) Argue that more generally, for the linear polarization state $\left|\phi\right\rangle$ (polarization angle ϕ), $\mathbf{W}_{1/2}(\theta)\left|\phi\right\rangle = \left|2\theta-\phi\right\rangle$.

Thus a half-wave plate can be used to rotate the angle of linearly polarized light. A *quarter-wave plate* is another birefringent optical element with a fast axis which may be rotated to any desired angle θ. Its operator $\mathbf{W}_{1/4}(\theta)$ again has eigenvectors $\left|\theta\right\rangle$ and $\left|\theta+90°\right\rangle$. Its representation is, however

$$\mathbf{W}_{1/4}(\theta) = \left|\theta\right\rangle\left\langle\theta\right| + \left|\theta+90°\right\rangle i\left\langle\theta+90°\right|.$$

$\mathbf{W}_{1/4}(\theta)$ shifts the phase of $\left|\theta+90°\right\rangle$ by $\pi/2$ relative to that of $\left|\theta\right\rangle$.

Problem 3:

(a) Show that $\mathbf{W}_{1/2}(\theta)\left|\psi\right\rangle = \left[\mathbf{W}_{1/4}(\theta)\right]^2\left|\psi\right\rangle$ for any polarization state vector $\left|\psi\right\rangle$.

(b) Show that $\mathbf{W}_{1/4}(45°)\left|V\right\rangle = \left|R\right\rangle$. Remember that for any state represented by a ket $\left|\psi\right\rangle$, $\left|\psi\right\rangle$ and $e^{i\phi}\left|\psi\right\rangle$ represent the same state (for any real number ϕ).